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# Singular Heteroclinic Cycles

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We study the unfolding of heteroclinic cycles of vector fields in $\mathbb{R}^n$, that possess a hyperbolic singularity and a saddle-node. The principal eigenvalues at the hyperbolic singularity are assumed to be real, but the weak hyperbolic eigenvalues at the saddle-node may be either real or complex conjugate. We discuss the bifurcation

tial expansions for the transition map near the saddle-node. 



## 1. INTRODUCTION

This paper contains a study of bifurcations from singular heteroclinic cycles in two parameter families of vector fields. The heteroclinic cycles consist of a hyperbolic singularity, a saddle-node and two heteroclinic orbits between them. The principal (i.e. weak stable and weak unstable) eigenvalues at the hyperbolic singularity are assumed to be real. We study bifurcations from those of these cycles that form a codimension two phenomenon, which means that they occur persistently in two parameter families of vector fields at isolated parameter values. Prototypes of such cycles occur in $\mathbb{R}^2$ and $\mathbb{R}^3$. In Fig. 1 a singular heteroclinic cycle in $\mathbb{R}^3$ is depicted, where at the saddle-node the linearized vector field possesses two complex conjugate stable eigenvalues. Vector fields in $\mathbb{R}^2$ can possess singular heteroclinic cycles where at the saddle-node the linearized vector field has a real stable eigenvalue (or unstable, that is equivalent under reversing the direction of the time parametrization). Both the hyperbolic singularity and the saddle-node singularity can possess additional strong stable and strong unstable directions, the dimensions of which are the same for the two singularities. Bifurcations from singular heteroclinic cycles in $\mathbb{R}^2$ were considered in [13].

Of particular interest is the bifurcation structure of heteroclinic cycles, where the weak stable eigenvalues at the saddle-node are complex



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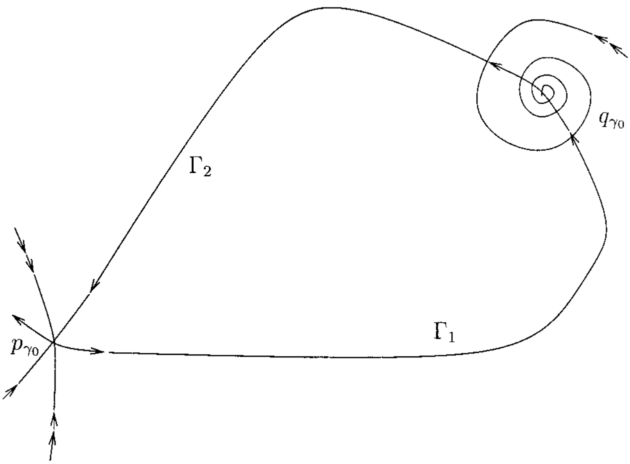



**FIG. 1.** A singular heteroclinic cycle of a family of vector fields $\{X(\cdot\,;\gamma)\}$, occurring at a parameter value $\gamma=\gamma_0$, consists of a hyperbolic singularity $p_{\gamma_0}$, a saddle-node $q_{\gamma_0}$ and heteroclinic connections $\Gamma_1$ from $p_{\gamma_0}$ to $q_{\gamma_0}$ and $\Gamma_2$ from $q_{\gamma_0}$ to $p_{\gamma_0}$. In this three dimensional picture, $DX(q_{\gamma_0};\gamma_0)$ has complex conjugate stable eigenvalues.

conjugate, like in Fig. 1. We will see that complex bifurcation structures can appear. The main feature that occurs in the unfolding is a cascade of inclination flip bifurcations. Depending on eigenvalue conditions this can lead to chaotic dynamics. The three dimensional case is discussed in [17].

The bifurcation analysis proceeds through a mixture of analytical techniques, such as the implicit mapping theorem and a Lyapunov–Schmidt reduction, and techniques of a more geometrical nature, such as the construction of invariant manifolds and foliations using graph transforms. We will give a glossary of the methods used in this paper. The detailed analysis is contained in Sections 3, 4 and 5. The resulting bifurcation theorems are contained in Section 2.

Consider a two parameter family of $n$-dimensional vector fields $\{X(\cdot\,;\gamma)\}$, $\gamma\in\mathbb{R}^2$, possessing a singular heteroclinic cycle for $\gamma=\gamma_0$. The precise conditions will be stated in Section 2. The bifurcation analysis is divided into different steps, which we will briefly discuss.

- A study of the Poincaré return map $\Pi$ on a cross section $\Sigma^{in}$, transverse to the heteroclinic orbit $\Gamma_2$ ($\Pi$ depends on the parameters, but in this brief overview we suppress this dependence from the notation). Near the hyperbolic singularity, one can take coordinates $(x_{ss}, x_s, x_u, x_{uu})$ that correspond to the splitting in strong unstable, weak unstable, weak stable, and strong stable directions. By assumption, the weak stable and unstable directions are both one dimensional. We may take $\Sigma^{in} = \{x_s = \delta\}$ for some small $\delta$. Write orbits of $\Pi$ as

$$\mathbf{x}_{j+1} = \Pi(\mathbf{x}_j), \tag{1.1}$$

where $\mathbf{x}_j = (x_{ss,\,j}, \delta, x_{u,\,j}, x_{uu,\,j})$.

- The derivation of bifurcation equations for periodic and homoclinic orbits. Of course, a periodic orbit is nothing but an orbit $\mathbf{x}_{j+1} = \Pi(\mathbf{x}_j)$ of $\Pi$ that satisfies $\mathbf{x}_N = \mathbf{x}_0$ for some positive integer $N$. We call this an $N$-periodic orbit. In principal, (1.1) with $\mathbf{x}_N = \mathbf{x}_0$ gives equations for $N$-periodic orbits. Caused by the existence of strong unstable directions, there don't seem to be convenient asymptotic expansions for $\Pi$ that would allow one to solve these equations. Instead, one obtains equations for the existence of $N$-periodic orbits from so-called Shil'nikov variables. In a nutshell, the equations $\mathbf{x}_{j+1} = \Pi(\mathbf{x}_j)$ and $\mathbf{x}_N = \mathbf{x}_0$ are replaced by equations of the form

$$\begin{aligned} x_{ss,\,j+1} &= G^{ss}(x_{ss,\,j}, x_{u,\,j}, x_{uu,\,j+1}), \\ x_{u,\,j+1} &= G^{u}(x_{ss,\,j}, x_{u,\,j}, x_{uu,\,j+1}), \\ x_{uu,\,j} &= G^{uu}(x_{ss,\,j}, x_{u,\,j}, x_{uu,\,j+1}) \end{aligned} \tag{1.2}$$

and $\mathbf{x}_N = \mathbf{x}_0$. Similar equations are derived for homoclinic orbits. The theory of Shil'nikov variables yields asymptotic expansions (in particular, determines the leading terms) for $G^{ss}$, $G^{u}$, $G^{uu}$, see Proposition 3.4.

- The equations (1.2) for $N$-periodic orbits provide $n-1$ (the dimension of $\Sigma^{in}$) times $N$ equations. By the Lyapunov–Schmidt method, one solves $x_{uu,\,j}$ and $x_{ss,\,j}$ as functions of $(x_{u,\,0}, ..., x_{u,\,N-1})$. Thus one obtains $N$ reduced bifurcation equations (Proposition 4.1). These can be solved for $N = 1, 2$, but become hard to manage for higher $N$. So one can solve for existence and bifurcations of $N$-periodic orbits, for $N = 1, 2$. Similarly for homoclinic orbits.

- By the construction of invariant (strong stable, strong unstable) foliations and invariant (center) manifolds (e.g. Proposition 5.1), one can in several cases exclude the existence of periodic and homoclinic orbits making more then two turns before closing. Also, the stability of a periodic orbit, or more generally the dimensions of its stable and unstable manifolds, is directly obtained from the existence of the invariant manifolds and foliations.

By restricting the flow to an invariant center manifold one obtains a dimension reduction of the problem. It should however be mentioned that the constructed center manifolds can not be used to study bifurcations of periodic orbits. For this purpose they lack sufficient smoothness; they are in general only once continuously differentiable. It is for this reason that we treat the bifurcations by a combination of analytical with geometrical techniques, as just sketched.

The Poincaré return map $\Pi$ is written as the composition of two transition maps through neighborhoods of the singularities. That is, one considers a second cross section $\Sigma^{out}$ that intersects $\Gamma_1$ transversally, and writes $\Pi$ as the composition of transition maps $\Pi_{loc}\colon \Sigma^{in} \to \Sigma^{out}$ and $\Pi_{far}\colon \Sigma^{out} \to \Sigma^{in}$. Note that the transition map $\Pi_{far}$ through a neighborhood of the saddle-node only makes sense for parameter values for which the saddle-node has disappeared and no singularities exist in its vicinity. Appropriate expressions for the transition maps $\Pi_{loc}$ and $\Pi_{far}$ are obtained from a study of Shil'nikov variables. They are combined to obtain asymptotic expressions for $\Pi$, in Shil'nikov variables. Shil'nikov variables, whose basic theory was developed in [32], [7], have been successfully applied in a large number of global bifurcation problems. A very readable account of the treatment of a global bifurcation problem using Shil'nikov variables is the paper [4] by S.-N. Chow, B. Deng and B. Fiedler.

While Shil'nikov variables near hyperbolic singularities are well studied, this is less so the case for Shil'nikov variables near nonhyperbolic singularities. In [31], [9] Shil'nikov variables near saddle-nodes are studied and used in the analysis of homoclinic bifurcations involving saddle-nodes. Compare further [34], [6], [24]. The estimates in [31], [9] do not suffice for the study of e.g. saddle-node bifurcations of periodic orbits or period doubling bifurcations, which are encountered in this paper. We derive asymptotic expansions for Shil'nikov coordinates that do suffice, applying ideas from [7], [8], [9]. Such expansions might also be useful in the study of other bifurcations in which saddle-nodes play a role; see e.g. the bifurcations in [3], [25], which have been treated under the assumption of nonresonance conditions among the hyperbolic eigenvalues.

Recall from our glossary that the bifurcation equations enable a study of $N$-periodic and $N$-homoclinic orbits for $N = 1, 2$; for higher $N$ the equations are much harder to treat. Indeed, for the resonant homoclinic bifurcation as studied in [4] and for the inclination-flip as studied in [22], no results on $N$-periodic or $N$-homoclinic orbits were obtained for $N \geqslant 3$. This can however be remedied by, in addition to solving the reduced bifurcation equations for $N = 1, 2$, constructing an invariant center manifold or invariant foliations. On a two dimensional center manifold, for instance, $N$-periodic and $N$-homoclinic orbits with $N \geqslant 3$ can not occur. For the bifurcation problem in [4], a center manifold near the homoclinic orbit was constructed in [29]. For the bifurcation problem studied in [22], for three dimensional vector fields, the nonexistence of $N$-periodic and $N$-homoclinic orbits for $N \geqslant 3$ was shown using invariant foliations [23], [19]. In our bifurcation problem, by constructing invariant manifolds and invariant foliations we exclude the existence of $N$-periodic and $N$-homoclinic orbits for $N \geqslant 3$ (in one of the occurring cases this does not work, we will argue that one can expect chaotic dynamics to occur in this case). As a byproduct, this also shows the nonexistence of $N$-periodic and $N$-homoclinic orbits for $N \geqslant 3$ in the unfolding of the inclination-flip as studied in [23], in any dimension.

The organization of the paper is as follows. In the next section we state the bifurcation theorems. Section 3 we provide asymptotic expansions for the transition maps, making use of Shil'nikov variables. The validity of these expansions is proved in the appendix. In Section 4 we derive bifurcation equations for bifurcations of periodic and homoclinic orbits. We solve them for $N$-periodic orbits and $N$-homoclinic orbits, with $N = 1, 2$. Section 5 contains the construction of invariant manifolds and foliations from which the absence of $N$-periodic and $N$-homoclinic orbits, in those bifurcations whose unfoldings do not show any complicated dynamics, follows.

## 2. BIFURCATION THEOREMS

The bifurcation theorems are stated in this section. First we list the conditions on the families we consider. Let $\{X(\cdot\,;\gamma)\}$ with $\gamma \in \mathbb{R}^2$ be a smooth two parameter family of vector fields on $\mathbb{R}^n$. By the adjective smooth we mean $C^\infty$. We remark though that the results also hold for $C^k$ families with $k$ large enough. We assume that $\{X(\cdot\,;\gamma)\}$ satisfies the following conditions.

> (HS : Hyperbolic singularity) For $\gamma$ near $\gamma_0$, the vector field $X(\cdot\,;\gamma)$ possesses a hyperbolic singularity $p_\gamma$ at which the linearization $DX(p_\gamma;\gamma)$ possesses one real weak unstable eigenvalue $\lambda^u(\gamma)$, $p^{uu}$ strong unstable eigenvalues $\lambda_j^{uu}(\gamma)$, one real weak stable eigenvalue $\lambda^s(\gamma)$ and $p^{ss} = n - p^{uu} - 2$ strong stable eigenvalues $\lambda_i^{ss}(\gamma)$. That is,

$$\operatorname{Re}\, \lambda_i^{ss}(\gamma) < \lambda^s(\gamma) < 0 < \lambda^u(\gamma) < \operatorname{Re}\, \lambda_j^{uu}(\gamma).$$

Denote by $W^{ss,s}(p_\gamma)$ the stable manifold of $p_\gamma$ and by $W^{u,uu}(p_\gamma)$ its unstable manifold. The stable manifold $W^{ss,s}(p_\gamma)$ is foliated by an invariant strong stable foliation $\mathfrak{G}^{ss}$ with $p^{ss}$ dimensional leaves. Similarly, there is an invariant strong unstable foliation $\mathfrak{G}^{uu}$ of $W^{u,uu}(p_\gamma)$.

There further are $(p^{uu}+2)$-dimensional center unstable manifolds $W^{s,u,uu}(p_\gamma)$. These manifolds are not unique and in general only of finite smoothness (in fact, they are $C^r$ for any $r < \min_j\{\operatorname{Re}\, \lambda_j^{uu}(\gamma)\}/\lambda^u(\gamma)$). Though nonunique, the tangent bundle of $W^{s,u,uu}(p_\gamma)$ along $W^{u,uu}(p_\gamma)$ is a uniquely determined smooth bundle. See e.g. [15], [16]. Similarly there exist $(p^{ss}+2)$-dimensional center stable manifolds $W^{ss,s,u}(p_\gamma)$. These manifolds are also not unique and of finite smoothness, and they possess a unique smooth tangent bundle along $W^{ss,s}(p_\gamma)$.

(NS : Nonhyperbolic singularity) At $\gamma=\gamma_0$, $X$ possesses a nonhyperbolic singularity $q_{\gamma_0}$ at which the the linearization $DX(q_{\gamma_0};\gamma_0)$ possesses one eigenvalue 0, either

(1) one real weak stable eigenvalue $\nu^s$, or

(2) two complex conjugate weak stable eigenvalues $\nu^s \pm i\omega^s$,

and further $q^{uu}=p^{uu}$ strong unstable eigenvalues $\nu_j^{uu}$ and $q^{ss}$ strong stable eigenvalues $\nu_i^{ss}$. So

$$\mathrm{Re}\ \nu_i^{ss} < \nu^s < 0 < \mathrm{Re}\ \nu_j^{uu}.$$

There is a coordinate $y_c$ on a center manifold of $q_{\gamma_0}$, in which $X(y;\gamma_0)$, restricted to the center manifold, is given by $(y_c^2+\mathcal{O}(y_c^3))(\partial/\partial y_c)$.

Note that $q^{ss}=p^{ss}-1$ if $DX(q_{\gamma_0};\gamma_0)$ has a real weak stable eigenvalue and $q^{ss}=p^{ss}-2$ if $DX(q_{\gamma_0};\gamma_0)$ has complex conjugate weak stable eigenvalues.

By the above conditions, the unstable set $W^{\mathrm{unst}}(q_{\gamma_0})$ of $q_{\gamma_0}$ is a smooth $(q^{uu}+1)$ dimensional manifold with boundary, the boundary is formed by the $q^{uu}$ dimensional strong unstable manifold $W^{uu}(q_{\gamma_0})$ of $q_{\gamma_0}$. Likewise, the stable set $W^{\mathrm{st}}(q_{\gamma_0})$ of $q_{\gamma_0}$ is a smooth manifold with boundary, the boundary is formed by the strong stable manifold $W^{ss,s}(q_{\gamma_0})$ of $q_{\gamma_0}$. For the two cases we consider, $W^{\mathrm{st}}(q_{\gamma_0})$ is either $(q_{ss}+2)$ or $(q_{ss}+3)$ dimensional.

There exists a smooth strong stable foliation $\mathfrak{F}^{ss}$ of $W^{\mathrm{st}}(q_{\gamma_0})$ with $q_{ss}$ dimensional leaves, as well as a strong stable foliation $\mathfrak{F}^{ss,s}$ of $W^{\mathrm{st}}(q_{\gamma_0})$ with leaves of codimension one in $W^{\mathrm{st}}(q_{\gamma_0})$. Similarly, there is a smooth strong unstable foliation $\mathfrak{F}^{uu}$ of $W^{\mathrm{unst}}(q_{\gamma_0})$ with $q_{uu}$ dimensional leaves.

There moreover exist center unstable manifolds $W^{s,c,uu}(q_{\gamma_0};\gamma_0)$ near $q_{\gamma_0}$ (we explicitly include the parameter value $\gamma_0$ in the notation, since later we will consider such manifolds for $\gamma$ close to $\gamma_0$). Although not unique and only of finite smoothness (in fact, they are $C^r$ for any $r<\min_i\{\mathrm{Re}\ \nu_i^{ss}\}/\nu^s$), one can show that they possess a unique smooth tangent bundle along $W^{\mathrm{unst}}(q_{\gamma_0})$ at $\gamma=\gamma_0$. This phenomenon is similar to center unstable manifolds near hyperbolic singularities.

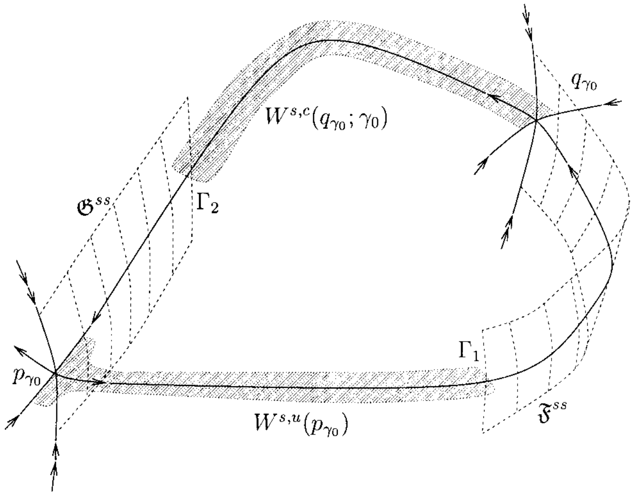



**FIG. 2.** This figure illustrates the containment and transversality conditions for a three dimensional vector field with a singular cycle, where at the saddle-node there is a real weak stable and a real strong stable eigenvalue.

(SC : Singular cycle) At $\gamma=\gamma_0$, $W^{\text{unst}}(q_{\gamma_0})$ intersects $W^{ss,\,s}(p_{\gamma_0})$ along an orbit $\Gamma_2$ and $W^{\text{st}}(q_{\gamma_0})$ intersects $W^{u,\,uu}(p_{\gamma_0})$ along an orbit $\Gamma_1$.

(CC : Containment conditions)

$$\Gamma_1 \not\subset W^{uu}(p_{\gamma_0}), \qquad \Gamma_1 \not\subset W^{ss,\,s}(q_{\gamma_0}),$$
$$\Gamma_2 \not\subset W^{ss}(p_{\gamma_0}), \qquad \Gamma_2 \not\subset W^{uu}(q_{\gamma_0}).$$

(TC : Transversality conditions)

$$\mathfrak{G}^{uu} \pitchfork_{\Gamma_1} W^{\text{st}}(q_{\gamma_0}), \qquad W^{ss,\,s,\,u}(p_{\gamma_0}) \pitchfork_{\Gamma_2} \mathfrak{F}^{uu},$$
$$\mathfrak{G}^{ss} \pitchfork_{\Gamma_2} W^{s,\,c,\,uu}(q_{\gamma_0};\gamma_0), \qquad W^{s,\,u,\,uu}(p_{\gamma_0}) \pitchfork_{\Gamma_1} \mathfrak{F}^{ss}.$$

In the above, e.g. $\mathfrak{G}^{uu} \pitchfork_{\Gamma_1} W^{\text{st}}(q_{\gamma_0})$ means that for each $x\in\Gamma_1$, $T_x W^{\text{st}}(q_{\gamma_0}) \oplus T_x \mathfrak{G}^{uu}_x = T_x \mathbb{R}^n$, where $\mathfrak{G}^{uu}_x$ denotes the leaf of $\mathfrak{G}^{uu}$ through $x$.

Note that $W^{\text{st}}(q_{\gamma_0})$ and $W^{u,\,uu}(p_{\gamma_0})$ intersect eachother transversally along $\Gamma_1$. By (TC) and (CC), the bundle $\mathbf{F}^{uu}$ of strong unstable directions over $\overline{\Gamma_1 \cup \Gamma_2}$,

$$\mathbf{F}^{uu} = \bigcup_{x\in\Gamma_1\cup p_{\gamma 0}} T_x \mathfrak{G}^{uu}_x \cup \bigcup_{y\in\Gamma_2\cup q_{\gamma 0}} T_y \mathfrak{F}^{uu}_y, \tag{2.3}$$

is a continuous bundle. Furthermore, if $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue, also the bundle $\mathbf{F}^{ss}$ of strong stable directions over $\overline{\Gamma_1 \cup \Gamma_2}$,

$$\mathbf{F}^{ss} = \bigcup_{x \in \Gamma_2 \cup p_{\gamma 0}} T_x \mathfrak{G}_x^{ss} \cup \bigcup_{y \in \Gamma_1 \cup q_{\gamma 0}} T_y \mathfrak{F}_y^{ss}, \tag{2.4}$$

is a continuous bundle.

(EC : Eigenvalue conditions) $\lambda^s(\gamma_0) + \lambda^u(\gamma_0) \neq 0$. If $DX(q_{\gamma_0}; \gamma_0)$ has complex conjugate weak stable eigenvalues, then

$$\frac{1}{2} < \frac{-\lambda^s(\gamma_0)}{\lambda^u(\gamma_0)} \qquad \text{and} \qquad \frac{\operatorname{Re} \lambda_i^{ss}(\gamma_0)}{\lambda^u(\gamma_0)} > 1$$

for all $i$.

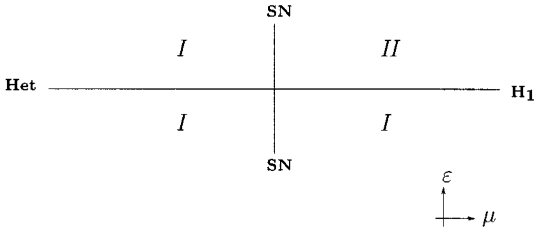
There remains an open set of eigenvalue conditions, for singular heteroclinic cycles where $DX(q_{\gamma_0}; \gamma_0)$ has complex conjugate weak stable eigenvalues, that we will not treat in detail. As will be discussed below, for these eigenvalue conditions we expect chaotic dynamics to occur.

(GU: Generic unfolding) The family $\{X(\cdot\,; \gamma)\}$ unfolds generically.

This condition means the following. The conditions (HS) to (EC) define a manifold, in the space of vector fields, of codimension two. The family $\{X(\cdot\,; \gamma)\}$ defines a two dimensional surface in the space of vector fields. Generic unfolding means that these two manifolds intersect transversally. Alternatively one may use the formulation that two functions of the parameters occurring naturally in the bifurcation study define a local submersion. These functions $(\mu, \varepsilon)$ are defined by (3.35), (4.48). All bifurcation diagrams below are depicted in the parameters $(\mu, \varepsilon)$.

We now state the bifurcation theorems. Different cases occur, depending on eigenvalue conditions and orientability of the bundles $\mathbf{F}^{ss}$, $\mathbf{F}^{uu}$ of strong stable and strong unstable directions over $\overline{\Gamma_1 \cup \Gamma_2}$.

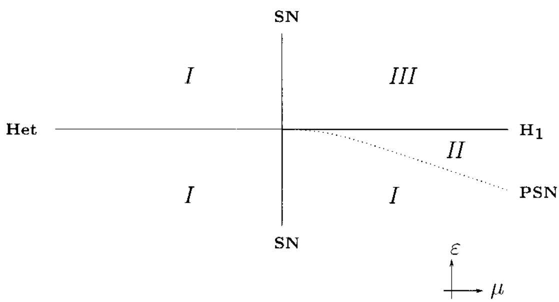
THEOREM 2.1. *Let $\{X(\cdot\,; \gamma)\}$ be a two parameter family of vector fields as above. Suppose that $\lambda^s(\gamma_0) + \lambda^u(\gamma_0) < 0$. After a reparametrization of the parameter plane, in new parameters $(\mu, \varepsilon)$ the bifurcation diagram of $\{X(\cdot\,; \gamma)\}$, is as depicted below.*




$\mathbf{H_1}$ *is a curve of* 1*-homoclinic orbits*, **SN** *is the curve of saddle-node bifurcations, and* **Het** *is a curve of heteroclinic orbits from a hyperbolic singularity near* $q_{\gamma_0}$ *to* $p_\gamma$.

*In region I,* $X(\,\cdot\,;\gamma)$ *has no periodic orbits. In region II,* $X(\,\cdot\,;\gamma)$ *has one periodic orbit*; *a* 1*-periodic orbit with* $(p^{ss}+2)$ *dimensional stable manifold and* $(p^{uu}+1)$ *dimensional unstable manifold.*

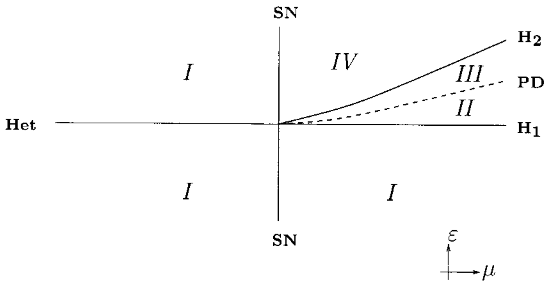
THEOREM 2.2. *Let* $\{X(\,\cdot\,;\gamma)\}$ *be a two parameter family of vector fields as above, with* $DX(q_{\gamma_0};\gamma_0)$ *possessing a real weak stable eigenvalue. Suppose that* $\lambda^s(\gamma_0)+\lambda^u(\gamma_0)>0$ *and that* $\mathbf{F}^{ss}\oplus\mathbf{F}^{uu}$ *is an orientable bundle over* $\overline{\Gamma_1\cup\Gamma_2}$. *After a reparametrization of the parameter plane, in new parameters* $(\mu,\varepsilon)$ *the bifurcation diagram of* $\{X(\,\cdot\,;\gamma)\}$, *is as depicted below.*




$\mathbf{H_1}$ *is a curve of* 1*-homoclinic orbits*, **SN** *is the curve of saddle-node bifurcations*, **Het** *is a curve of heteroclinic orbits from a hyperbolic singularity near* $q_{\gamma_0}$ *to* $p_\gamma$, *and* **PSN** *is a curve of periodic saddle-node bifurcations.*

*In region I,* $X(\,\cdot\,;\gamma)$ *has no periodic orbits. In region II,* $X(\,\cdot\,;\gamma)$ *has two periodic orbits*; *a* 1*-periodic orbit with* $(p^{ss}+2)$ *dimensional stable manifold and* $(p^{uu}+1)$ *dimensional unstable manifold and a* 1*-periodic orbit with* $(p^{ss}+1)$ *dimensional stable manifold and* $(p^{uu}+2)$ *dimensional unstable manifold. In region III,* $X(\,\cdot\,;\gamma)$ *has one periodic orbit*; *a* 1*-periodic orbit with* $(p^{ss}+1)$ *dimensional stable manifold and* $(p^{uu}+2)$ *dimensional unstable manifold.*

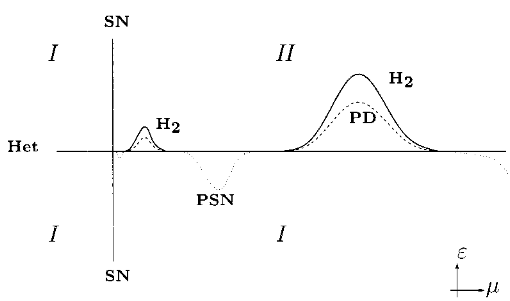
THEOREM 2.3. *Let $\{X(\cdot;\gamma)\}$ be a two parameter family of vector fields as above, with $DX(q_{\gamma_0};\gamma_0)$ possessing a real weak stable eigenvalue. Suppose that $\lambda^s(\gamma_0)+\lambda^u(\gamma_0)>0$ and that $\mathbf{F}^{ss}\oplus\mathbf{F}^{uu}$ is a nonorientable bundle over $\overline{\Gamma_1\cup\Gamma_2}$. After a reparametrization of the parameter plane, in new parameters $(\mu,\varepsilon)$ the bifurcation diagram of $\{X(\cdot;\gamma)\}$, is as depicted below.*




$\mathbf{H_1}$ *is a curve of* 1*-homoclinic orbits,* **SN** *is the curve of saddle-node bifurcations,* **Het** *is a curve of heteroclinic orbits from a hyperbolic singularity near $q_{\gamma_0}$ to $p_\gamma$,* **PD** *is a curve of period doubling bifurcations, and $\mathbf{H_2}$ is a curve of* 2*-homoclinic orbits.*

*In region I, $X(\cdot;\gamma)$ has no periodic orbits. In region II, $X(\cdot;\gamma)$ has one periodic orbit; a* 1*-periodic orbit with $(p^{ss}+1)$ dimensional stable manifold and $(p^{uu}+2)$ dimensional unstable manifold. In region III, $X(\cdot;\gamma)$ has two periodic orbits; a* 1*-periodic orbit with $(p^{ss}+2)$ dimensional stable manifold and $(p^{uu}+1)$ dimensional unstable manifold and a* 2*-periodic orbit with $(p^{ss}+1)$ dimensional stable manifold and $(p^{uu}+2)$ dimensional unstable manifold. In region IV, $X(\cdot;\gamma)$ has one periodic orbit; a* 1*-periodic orbit with $(p^{ss}+2)$ dimensional stable manifold and $(p^{uu}+1)$ dimensional unstable manifold.*

THEOREM 2.4. *Let $\{X(\cdot;\gamma)\}$ be a two parameter family of vector fields as above, with $DX(q_{\gamma_0};\gamma_0)$ possessing complex conjugate weak stable eigenvalues. Suppose that $-\lambda^{ss}(\gamma_0)/\lambda^u(\gamma_0)>1$ and $\frac{1}{2}<-\lambda^s(\gamma_0)/\lambda^u(\gamma_0)<1$. After a reparametrization of the parameter plane, in new parameters $(\mu,\varepsilon)$ the bifurcation diagram of $\{X(\cdot;\gamma)\}$, is as depicted below.*



*$\mathbf{H_1}$ is a curve of 1-homoclinic orbits, **SN** is the curve of saddle-node bifurcations, **Het** is a curve of heteroclinic orbits from a hyperbolic singularity near $q_{\gamma_0}$ to $p_\gamma$, **PD** are curves of period doubling bifurcations, $\mathbf{H_2}$ are curves of 2-homoclinic orbits, and **PSN** are curves of periodic saddle-node bifurcations. There is a sequence of inclination flip bifurcation points on $\mathbf{H_1}$, accumulating on the codimension two heteroclinic bifurcation point. From each of these points, curves **PD**, $\mathbf{H_2}$, and **PSN** branch.*

*In region I, $X(\cdot;\gamma)$ has no periodic orbits. In region II, $X(\cdot;\gamma)$ has one periodic orbit: a 1-periodic orbit with $(p^{ss}+3)$ dimensional stable manifold and $(p^{uu}+1)$ dimensional unstable manifold. In the regions bounded by the curves $\mathbf{H_1}$ and **PSN**, $X(\cdot;\gamma)$ has two periodic orbits: a 1-periodic orbit with $(p^{ss}+3)$ dimensional stable manifold and $(p^{uu}+1)$ dimensional unstable manifold and a 1-periodic orbit with $(p^{ss}+2)$ dimensional stable manifold and $(p^{uu}+2)$ dimensional unstable manifold. In the regions bounded by the curves $\mathbf{H_1}$ and **PD**, $X(\cdot;\gamma)$ has one periodic orbit: a 1-periodic orbit with $(p^{ss}+2)$ dimensional stable manifold and $(p^{uu}+2)$ dimensional unstable manifold. In the regions bounded by the curves **PD** and $\mathbf{H_2}$, $X(\cdot;\gamma)$ has two periodic orbits: a 1-periodic orbit with $(p^{ss}+3)$ dimensional stable manifold and $(p^{uu}+1)$ dimensional unstable manifold and a 2-periodic orbit with $(p^{ss}+2)$ dimensional stable manifold and $(p^{uu}+2)$ dimensional unstable manifold.*

The bifurcation diagram in the above theorem is somewhat reminiscent of the bifurcation structures of certain figure-eight homoclinic bifurcations and heteroclinic bifurcations with complex conjugate principal eigenvalues [14], [36], [30]. Remains the bifurcation as in Theorem 2.4 above, but with eigenvalue conditions $-\max_i\{\mathrm{Re}\ \lambda_i^{ss}(\gamma_0)\}/\lambda^u(\gamma_0)<1$ or $\frac{1}{2}> -\lambda^s(\gamma_0)/\lambda^u(\gamma_0)$. As in the proof of Theorem 2.4 one shows the existence of the bifurcation curves $\mathrm{H}_1$, Het and SN. Furthermore there is a sequence of inclination flip bifurcations on the curve $\mathrm{H}_1$, accumulating on the codimension

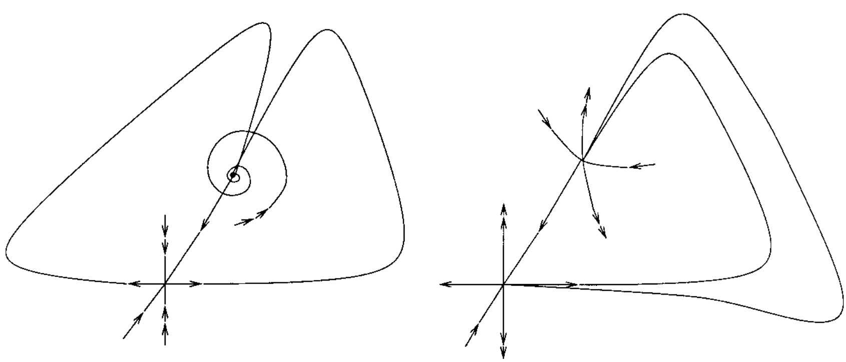


**FIG. 3.** Examples of vector fields with more then one singular heteroclinic cycle.

two heteroclinic bifurcation point. The unfolding of these inclination flips involves complicated dynamics, see [18], [26], [27], [20].

Finally we mention that the vector field $X_{\gamma_0}$ can possess several singular heteroclinic cycles; there can be more then one connection from $p_{\gamma_0}$ to $q_{\gamma_0}$. Indeed, the manifolds $W^{st}(q_{\gamma_0})$ and $W^{u,uu}(p_{\gamma_0})$ can intersect eachother transversally along several orbits. R. Bamon informed me that geometric Lorenz attractors can appear in unfoldings of three dimensional vector fields with two singular heteroclinic connections as in the left hand side of Fig. 3, see [17]. Another possible case is also depicted in Fig. 3; two orbits in the same component of $W^{u,uu}(p_{\gamma_0})\backslash W^{uu}(p_{\gamma_0})$ connect $p_{\gamma_0}$ to $q_{\gamma_0}$. Applying [16] one can show the existence of suspensions of horseshoes in its unfolding.

## 3. TRANSITION MAPS

In this section we provide exponential expansions for the transition maps, in Shil'nikov variables, near the singularities. Combining these gives exponential expansions for a Poincaré return map. First Shil'nikov variables for a transition map through a neighborhood of the hyperbolic singularity $p_\gamma$ are discussed, see Proposition 3.1. This is largely standard, although we state the results somewhat differing from the presentation in e.g. [7]. Secondly, Shil'nikov variables for a transition map through a neighborhood of the saddle-node $q_{\gamma_0}$ are discussed (for parameter values, where the saddle-node has disappeared), see Proposition 3.2.

What is obtained is the following. Given cross sections $\Sigma^{in}$ and $\Sigma^{out}$ transverse to $\Gamma_2$ and $\Gamma_1$ respectively, we have asymptotic expressions for the transition maps $\Pi_{loc}\colon \Sigma^{in} \to \Sigma^{out}$ and $\Pi_{far}\colon \Sigma^{out} \to \Sigma^{in}$, in Shil'nikov variables. Combining these gives asymptotic expressions for $\Pi = \Pi_{far} \circ \Pi_{loc}$. The asymptotic expressions given for the transition maps $\Pi_{loc}$ and $\Pi_{far}$ are valid only in suitable smooth coordinates. Therefore, we obtain two different coordinate systems on $\Sigma^{in}$ and two different coordinate systems on $\Sigma^{out}$. In combining the two asymptotic expansions, we must go from one coordinate system to the other, on both the sections $\Sigma^{in}$ and $\Sigma^{out}$. By controlling the freedom in choosing the coordinates required in studying Shil'nikov variables, we can take these coordinate changes to be affine coordinate changes. This makes it easy to combine the results on the two transition maps to get asymptotic expansions for the Poincaré return map, as discussed in Proposition 3.4.

### 3.1. *Near the Hyperbolic Singularity*

Take smooth, parameter dependent coordinates $x = (x_{ss}, x_s, x_u, x_{uu})$ on a small neighborhood $\mathcal{U}$ of the hyperbolic singularity $p_\gamma$ so that $p_\gamma$ is the origin $(0, 0, 0, 0)$ and

$$DX(p_\gamma;\gamma)=$$

$$A^{ss}x_{ss}\,\frac{\partial}{\partial x_{ss}}+\lambda^s x_s\,\frac{\partial}{\partial x_s}+\lambda^u x_u\,\frac{\partial}{\partial x_u}+A^{uu}x_{uu}\,\frac{\partial}{\partial x_{uu}}. \tag{3.5}$$

This is the splitting in strong stable coordinates $x_{ss}$, a weak stable coordinate $x_s$, a weak unstable coordinate $x_u$ and strong unstable coordinates $x_{uu}$. Here $A^{ss}$, $\lambda^s$, $\lambda^u$, $A^{uu}$ depend on the parameters, but we suppress this dependence from the notation. We will often use shorthand notation by grouping indices, for instance we will write $x_{ss,\,s}$ for $(x_{ss}, x_s)$. Let

$$\lambda^{ss}=\max_i\,\{\operatorname{Re}\lambda_i^{ss}\},$$
$$\lambda^{uu}=\min_j\,\{\operatorname{Re}\lambda_j^{uu}\}.$$

Take cross sections $\Sigma^{in}$ and $\Sigma^{out}$ near $p_\gamma$,

$$\Sigma^{in}=\{x_s=\delta,\ \|x_{ss}\|,\ |x_u|,\ \|x_{uu}\|\leqslant\delta\}, \tag{3.6}$$

$$\Sigma^{out}=\{x_u=\delta,\ \|x_{ss}\|,\ |x_s|,\ \|x_{uu}\|\leqslant\delta\}, \tag{3.7}$$

for some small $\delta>0$. By a rescaling we may assume $\delta=1$. Write $x^{in}=(x_{ss}^{in}, x_u^{in}, x_{uu}^{in})$ for the coordinate system on $\Sigma^{in}$ obtained by restricting the above coordinates near $p_\gamma$ to $\Sigma^{in}$. Likewise, define coordinates $x^{out}=(x_{ss}^{out}, x_s^{out}, x_{uu}^{out})$ on $\Sigma^{out}$.

The local transition map $\Pi_{loc}\colon \Sigma^{in}\to\Sigma^{out}$ assigns a point $x^{out}\in\Sigma^{out}$ to a point $x^{in}\in\Sigma^{in}$. The following proposition relates these points. This proposition is a corollary of results by [33], [28], [7], [8], see also [23].

**Proposition 3.1.** *The coordinates $x=(x_{ss}, x_s, x_u, x_{uu})$ near $p_\gamma$ can be chosen so that the following holds. Let $\Pi_{loc}\colon\Sigma^{in}\to\Sigma^{out}$ be the local transition map*

$$\Pi_{loc}(x_{ss}^{in}, x_u^{in}, x_{uu}^{in};\gamma)=(x_{ss}^{out}, x_s^{out}, x_{uu}^{out}).$$

*The coordinates $(x_{ss}^{out}, x_s^{out}, x_{uu}^{in})$ can be written as functions of $(x_{ss}^{in}, x_u^{in}, x_{uu}^{out})$. Let* $\max\{\lambda^{ss}, 2\lambda^s\}<\bar\lambda^{ss}<0$, $0<\bar\lambda^{uu}<\min\{\lambda^{uu}, 2\lambda^u\}$ *and write* $\beta=-\lambda^s/\lambda^u$.

$$x_{ss}^{out}=(x_u^{in})^{-\bar\lambda^{ss}/\lambda^u}R^{ss}(x_{ss,\,u}^{in}, x_{uu}^{out};\gamma),$$

$$x_s^{out}=(x_u^{in})^{\beta}(\psi^s(x_{ss}^{in}, x_{uu}^{out};\gamma)+R^s(x_{ss,\,u}^{in}, x_{uu}^{out};\gamma)),$$

$$x_{uu}^{in}=(x_u^{in})^{\bar\lambda^{uu}/\lambda^u}R^{uu}(x_{ss,\,u}^{in}, x_{uu}^{out};\gamma).$$

*Here $\psi^s$ is a smooth map satisfying $\psi^s(0, 0; \gamma) \neq 0$. Furthermore, $R^{ss}$, $R^s$, $R^{uu}$ are smooth for $x_u^{in} > 0$; there exist $\omega > 0$, $C_{k+l} > 0$ so that, for $i = ss, s, uu$,*

$$\left\| D^l \frac{\partial^k}{\partial (x_u^{in})^k} R^i(x_{ss, u}^{in}, x_{uu}^{out}; \gamma) \right\| \leqslant C_{k+l}(x_u^{in})^{\omega - k},$$

*where $D^l$ stands for the $l^{\text{th}}$ order derivative in $(x_{ss}^{in}, x_{uu}^{out}, \gamma)$.*

*Remark.* In the proof we explicitly list the properties the coordinates $x$ must fulfill (namely (3.8), ..., (3.11)). The coordinates $x$ can be chosen near compact parts of $p_{\gamma_0} \cup \Gamma_1 \cup \Gamma_2$, satisfying properties (3.8), ..., (3.11) and with the above proposition still valid. In particular, the cross sections $\Sigma^{in}$ and $\Sigma^{out}$ need not be close to $p_\gamma$.

*Proof.* Take coordinates $x$ near $p_\gamma$ so that $DX(p_\gamma; \gamma)$ is as in (3.5). By a smooth coordinate change, we may assume that

$$W^{ss, s}(p_\gamma) = \{x_{u, uu} = 0\}, \tag{3.8}$$

$$W^{u, uu}(p_\gamma) = \{x_{ss, s} = 0\}, \tag{3.9}$$

$$TW^{ss, s, u}(p_\gamma)|_{W^{ss, s}(p_\gamma)} = \{x_{uu} = 0\}, \tag{3.10}$$

$$TW^{s, u, uu}(p_\gamma)|_{W^{u, uu}(p_\gamma)} = \{x_{ss} = 0\}. \tag{3.11}$$

In such coordinates, $X$ has an expression

$$X(x; \gamma) = (A^{ss} x_{ss} + G^{ss}(x; \gamma)) \frac{\partial}{\partial x_{ss}} + (\lambda^s x_s + G^s(x; \gamma)) \frac{\partial}{\partial x_s}$$
$$+ (\lambda^u x_u + G^u(x; \gamma)) \frac{\partial}{\partial x_u} + (A^{uu} x_{uu} + G^{uu}(x; \gamma)) \frac{\partial}{\partial x_{uu}},$$

with

$$G^{ss}(x; \gamma) = \mathcal{O}(\|x_{ss}\| \, \|x\| + |x_s|^2), \tag{3.12}$$

$$G^s(x; \gamma) = \mathcal{O}(\|x_{ss, s}\| \, \|x\|), \tag{3.13}$$

$$G^u(x; \gamma) = \mathcal{O}(\|x_{u, uu}\| \, \|x\|), \tag{3.14}$$

$$G^{uu}(x; \gamma) = \mathcal{O}(\|x_{uu}\| \, \|x\| + |x_u|^2). \tag{3.15}$$

Indeed, (3.13) is a consequence of (3.9) and (3.14) is a consequence of (3.8). We claim that (3.12) is a consequence of (3.9) and (3.11). By (3.9), $G^{ss}$ is of the order $\mathcal{O}(\|x_{ss, s}\| \, \|x\|)$ for small $x$. The appearance of the term $|x_s|^2$, i.e. the absence of terms proportional to $|x_s| \|x_{u, uu}\|$, is due to (3.11) (this is what (3.11) means when stated in coordinates). Similarly, (3.15) is a consequence of (3.8) and (3.10).

Let $E^{ss} \times E^{s} \times E^{u} \times E^{uu}$ be the splitting of $\mathbb{R}^n$, corresponding to the coordinates $x=(x_{ss}, x_s, x_u, x_{uu})$. For $\tau>0$, $\xi_{ss} \in E^{ss}$, $\xi_{uu} \in E^{uu}$, let $x(t, \tau, \xi_{ss}, \xi_{uu}; \gamma)$ be the orbit of $X(\cdot\,; \gamma)$ with

$$\begin{aligned} x_{ss,\,s}(0, \tau, \xi_{ss}, \xi_{uu}; \gamma) &= (\xi_{ss}, 1), \\ x_{u,\,uu}(\tau, \tau, \xi_{ss}, \xi_{uu}; \gamma) &= (1, \xi_{uu}). \end{aligned} \tag{3.17}$$

By [32], [7], if $\|\xi_{ss}\| \leqslant 1$, $\|\xi_{uu}\| \leqslant 1$, $\delta$ small enough and $\tau$ large enough, there is a unique orbit satisfying these boundary conditions. Note that $x(0, \tau, \xi_{ss}, \xi_{uu}; \gamma) \in \Sigma^{in}$ and $x(\tau, \tau, \xi_{ss}, \xi_{uu}; \gamma) \in \Sigma^{out}$, so that $\tau$ equals the transition time of the orbit $x(t, \tau, \xi_{ss}, \xi_{uu}; \gamma)$ between $\Sigma^{in}$ and $\Sigma^{out}$. Moreover, it follows as in the appendix (ignoring the additional center coordinate that is considered there), or as in [7], [8], that the following asymptotics hold. Writing $r=e^{-\lambda^u \tau}$, one has

$$x_{ss}^{out} = r^{-\bar{\lambda}^{ss}/\lambda^u} T^{ss}(r, x_{ss}^{in}, x_{uu}^{out}; \gamma), \tag{3.16}$$

$$x_{s}^{out} = r^{\beta}(\phi^{s}(x_{ss}^{in}, x_{uu}^{out}; \gamma) + T^{s}(r, x_{ss}^{in}, x_{uu}^{out}; \gamma)), \tag{3.17}$$

$$x_{u}^{in} = r(\phi^{u}(x_{ss}^{in}, x_{uu}^{out}; \gamma) + T^{u}(r, x_{ss}^{in}, x_{uu}^{out}; \gamma)), \tag{3.18}$$

$$x_{uu}^{in} = r^{\bar{\lambda}^{uu}/\lambda^u} T^{uu}(r, x_{ss}^{in}, x_{uu}^{out}; \gamma). \tag{3.19}$$

Here $\phi^s, \phi^u$ are smooth. Furthermore, $T^{ss}, T^{s}, T^{uu}$ are smooth for $r>0$. There exist $\omega>0$, $C_{k+l}>0$ so that, for $i = ss, s, u, uu$,

$$\left\| D^l \frac{\partial^k}{\partial r^k} T^i(r, x_{ss}^{in}, x_{uu}^{out}; \gamma) \right\| \leqslant C_{k+l} r^{\omega - k}, \tag{3.20}$$

where $D^l$ stands for the $l^{\text{th}}$ order derivative in $(x_{ss}^{in}, x_{uu}^{out}, \gamma)$.

By the implicit function theorem, one can solve $r$ as function of $x_u^{in}$, $x_{ss}^{in}$, $x_{uu}^{out}$ and $\gamma$ from (3.18). Note that $r$ is a smooth function for $x_u>0$. Estimates on derivatives follow from the implicit function theorem, by differentiating (3.18) and using the estimates (3.20). Putting $r$ as function of $x_u^{in}$, $x_{ss}^{in}$, $x_{uu}^{out}$ and $\gamma$ in the equations (3.16), (3.17) and (3.19), proves Proposition 3.1. ▮

### 3.2. *Near the Saddle-Node Singularity*

Take smooth, parameter dependent coordinates $y=(y_{ss}, y_s, y_c, y_{uu})$ on a small neighborhood $\mathscr{U}$ of the saddle-node singularity $q_{\gamma_0}$ so that, at $\gamma=\gamma_0$, $q_{\gamma_0}$ equals the origin $(0,0,0,0)$ and

$$DX(q_{\gamma_0}; \gamma_0) = B^{ss} y_{ss} \frac{\partial}{\partial y_{ss}} + B^{s} y_{s} \frac{\partial}{\partial y_{s}} + B^{uu} y_{uu} \frac{\partial}{\partial y_{uu}}. \tag{3.21}$$

Here $y_s$ is a one dimensional coordinate and $B^s = \nu^s$ is case $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue $\nu^s$. And, if $DX(q_{\gamma_0}; \gamma_0)$ has two complex conjugate eigenvalues $\nu^s \pm i\omega^s$, then $y_s$ is a two dimensional coordinate and $B^s$ is a $2 \times 2$ matrix with $\nu^s \pm i\omega^s$ as its eigenvalues. The coordinates $y_{ss}$ and $y_{uu}$ are the strong stable and strong unstable coordinates, respectively. We suppress the dependence of $B^{ss}$, $B^s$, $B^{uu}$ on the parameters. We recall that we will often shorten notation by writing e.g. $y_{ss,\,s}$ for $(y_{ss}, y_s)$. Let

$$\nu^{ss} = \max_i \{\operatorname{Re} \nu_i^{ss}\},$$

$$\nu^{uu} = \min_j \{\operatorname{Re} \nu_j^{uu}\}.$$

Consider cross sections $S^{in}$ and $S^{out}$, transverse to $\Gamma_1$ and $\Gamma_2$, respectively, of the form

$$S^{in} = \{y_c = -1, \|y_{ss}\|, |y_s|, \|y_{uu}\| \leqslant \delta\}, \tag{3.22}$$

$$S^{out} = \{y_c = 1, \|y_{ss}\|, |y_s|, \|y_{uu}\| \leqslant \delta\}, \tag{3.23}$$

for some small $\delta > 0$. Write $y^{in} = (y_{ss}^{in}, y_s^{in}, y_{uu}^{in})$ for the coordinate system on $S^{in}$ obtained by restricting the above coordinates near $p_\gamma$ to $S^{in}$. Likewise, define coordinates $y^{out} = (y_{ss}^{out}, y_s^{out}, y_{uu}^{out})$ on $S^{out}$. For parameter values $\gamma$ for which $X(\cdot\,; \gamma)$ has no singularities near $q_{\gamma_0}$, the local transition map $\Phi_{loc}\colon S^{in} \to S^{out}$ is defined. In (3.35) below a function $\mu(\gamma)$ is defined, so that $\{X \cdot\,; \gamma)$ has a saddle-node if $\mu(\gamma) = 0$ (so $\mu(\gamma_0) = 0$) and no singularities near $q_{\gamma_0}$ if $\mu(\gamma) > 0$. The following proposition provides asymptotic expansions of $\Phi_{loc}$.

Proposition 3.2. *The coordinates $y = (y_{ss}, y_s, y_c, y_{uu})$ near $q_{\gamma_0}$ can be chosen so that the following is true. Consider values of $\gamma$ near $\gamma_0$ for which $\mu(\gamma) > 0$. Let $\Phi_{loc}\colon S^{in} \to S^{out}$ be the local transition map*

$$\Phi_{loc}(y_{ss,\,s}^{in}, y_{uu}^{in}; \gamma) = (y_{ss,\,s}^{out}, y_{uu}^{out}).$$

*The coordinates $(y_{ss,\,s}^{out}, y_{uu}^{in})$ can be written as functions of $(y_{ss,\,s}^{in}, y_{uu}^{out})$. Fix $k > 0$, $\max\{2\nu^s, \nu^{ss}\} < \bar{\nu}^{ss} < 0$ and $\bar{\nu}^{uu} < \nu^{uu}$. Let $(y_{ss,\,s}^{in}, y_{uu}^{out}, \gamma) \mapsto \hat{\tau}(y_{ss,\,s}^{in}, y_{uu}^{out}; \gamma)$ be the function that gives the passage time of the orbit between $(y_{ss,\,s}^{in}, y_{uu}^{in})$ and $(y_{ss,\,s}^{out}, y_{uu}^{out})$. Write $\rho = e^{-\hat{\tau}(0,\,0;\,\gamma)}$. Then*

$$y_{ss}^{out} = \rho^{\bar{\nu}^{ss}} U^{ss}(y_{ss,\,s}^{in}, y_{uu}^{out}; \gamma),$$

$$y_s^{out} = \rho^{\nu^s} e^{R(\gamma)} (\psi^s(y_{ss,\,s}^{in}, y_{uu}^{out}; \gamma) + U^s(y_{ss,\,s}^{in}, y_{uu}^{out}; \gamma)),$$

$$y_{uu}^{in} = \rho^{\bar{\nu}^{uu}} U^{uu}(y_{ss,\,s}^{in}, y_{uu}^{out}; \gamma).$$

*The function $\rho$ is smooth; $\rho$ and its derivatives are flat functions as $\mu(\gamma) \to 0$. The function $\psi^s$ is smooth; $\psi^s(0, y^{out}_{uu}; \gamma) = 0$ and $(\partial/\partial y^{in}_s)\, \psi^s(0, 0; \gamma)$ is invertible. If $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue, then*

$$R(\gamma) = 0.$$

*If $DX(q_{\gamma_0}; \gamma_0)$ has complex conjugate weak stable eigenvalues, then $R(\gamma)$ is a matrix of the form*

$$R(\gamma) = \begin{pmatrix} 0 & -\Omega(\gamma) \\ \Omega(\gamma) & 0 \end{pmatrix},$$

*where $\Omega(\gamma)/\tau(0, 0; \gamma) \to \omega^s$ and $\|(\partial/\partial\gamma)\, \Omega(\gamma)\| \to \infty$ as $\mu(\gamma) \to 0$.*

*The maps $U^{ss,s}$, $U^{uu}$ are smooth and satisfy $U^{ss,s}(0, y^{out}_{uu}; \gamma) = 0$ and $U^{uu}(y^{in}_{ss,s}, 0; \gamma) = 0$. Furthermore, for $i = ss, s, uu$ and for some $\omega > 0$, $C_l > 0$,*

$$\|D^l U^i(y^{in}_{ss,s}, y^{out}_{uu}; \gamma)\| \leqslant C_l \rho^{\omega}. \tag{3.31}$$

*Here $D^l$ stands for the $l^{\text{th}}$ order derivative in $(y^{in}_{ss,s}, y^{out}_{uu}, \gamma)$.*

*Remark.* In the proof we explicitly list the properties the coordinates $y$ must fulfill (namely (3.24), ..., (3.28) and (3.34)). We can extend these coordinates, with the same properties, to a small neighborhood of a compact part of the center manifold of $q_{\gamma_0}$ that contains $\Gamma_1$ and $\Gamma_2$; the cross sections $S^{in}$ and $S^{out}$ need not be chosen close to $q_{\gamma_0}$. In fact, we can choose

$$S^{in} = \Sigma^{out},$$
$$S^{out} = \Sigma^{in},$$

where $\Sigma^{in}$ and $\Sigma^{out}$ are defined by (3.6) and (3.7). The expansions in the above proposition remain valid.

Before proving the proposition, we recall some facts about certain invariant manifolds near $q_{\gamma_0}$ and their smoothness properties. See e.g. [15] for more information on invariant manifolds. As mentioned in Section 2, for $\gamma = \gamma_0$ the stable set $W^{\text{st}}(q_{\gamma_0})$ of $q_{\gamma_0}$ forms a smooth manifold with boundary. One can extend $W^{\text{st}}(q_{\gamma_0})$ to a smooth center stable manifold $W^{ss,s,c}(q_{\gamma_0}; \gamma_0)$. This manifold is not unique. It is however persistent; for each positive integer $k$, there is a neighborhood $\mathscr{V}_k$ of $\gamma_0$ in the parameter space, so that for $\gamma \in \mathscr{V}_k$, there is a $C^k$ invariant center stable manifold $W^{ss,s,c}(q_{\gamma_0}; \gamma)$, which also depends $C^k$ on $\gamma$. The loss of smoothness takes only place along hyperbolic singularities and their stable and unstable manifolds, in particular only for $\gamma$ with $\mu(\gamma) < 0$. By restricting $\gamma$ to $\mathscr{V}_k \cap \{\mu(\gamma) \geqslant 0\}$, the manifolds $W^{ss,s,c}(q_{\gamma_0}; \gamma)$ can be chosen to be smooth and depending smoothly on $\gamma$. Also, $W^{ss,s,c}(q_{\gamma_0}; \gamma)$ supports a smooth strong stable foliation $\mathfrak{F}^{ss}$. The same remarks apply to center unstable manifolds $W^{c,uu}(q_{\gamma_0};\gamma)$ that extend the unstable set $W^{\mathrm{unst}}(q_{\gamma_0})$. They support a smooth strong unstable foliation $\mathfrak{F}^{uu}$. We will write

$$W^c(q_{\gamma_0};\gamma)=W^{ss,s,c}(q_{\gamma_0};\gamma)\cap W^{c,uu}(q_{\gamma_0};\gamma).$$

*Proof of Proposition* 3.2. By the preceding considerations, there is a smooth coordinate change so that

$$W^{ss,s,c}(q_{\gamma_0};\gamma)=\{y_{uu}=0\},\tag{3.24}$$

$$W^{c,uu}(q_{\gamma_0};\gamma)=\{y_{ss,s}=0\},\tag{3.25}$$

$$TW^{s,c,uu}(q_{\gamma_0};\gamma)|_{W^{c,uu}(q_{\gamma 0})}=\{y_{ss}=0\},\tag{3.26}$$

$$\mathfrak{F}^{ss,s}|_{W^{ss,s,c}(q_{\gamma 0};\gamma)}=\{y_c=\text{const.},\ y_{uu}=0\},\tag{3.27}$$

$$\mathfrak{F}^{uu}|_{W^{c,uu}(q_{\gamma 0};\gamma)}=\{y_{ss,s}=0,\ y_c=\text{const.}\}.\tag{3.28}$$

This yields

$$\begin{aligned}X(y;\gamma)&=(B^{ss}y_{ss}+F^{ss}(y;\gamma))\frac{\partial}{\partial y_{ss}}+(B^s(y_c;\gamma)\,y_s+F^s(y;\gamma))\frac{\partial}{\partial y_s}\\&\quad+(U^c(y_c;\gamma)+F^c(y;\gamma))\frac{\partial}{\partial y_c}+(B^{uu}y_{uu}+F^{uu}(y;\gamma))\frac{\partial}{\partial y_{uu}},\end{aligned}\tag{3.29}$$

with

$$F^{ss}(y;\gamma)=\mathcal{O}(\|y_{ss}\|\ \|y\|+\|y_s\|^2),\tag{3.30}$$

$$F^s(y;\gamma)=\mathcal{O}(\|y_{ss}\|\ \|y\|+\|y_s\|\ \|y_{ss,s,uu}\|),\tag{3.31}$$

$$F^c(y;\gamma)=\mathcal{O}(\|y_{ss,s}\|\ \|y_{uu}\|),\tag{3.32}$$

$$F^{uu}(y;\gamma)=\mathcal{O}(\|y_{uu}\|\ \|y\|).\tag{3.33}$$

Indeed, (3.30) is a consequence of (3.25) and (3.26), (3.31) is a consequence of (3.25), (3.32) is a consequence of (3.24), (3.25), (3.27), (3.28) and (3.33) is a consequence of (3.24). Compare the proof of Proposition 3.1.

Note that (3.24) and (3.25) imply

$$W^c(q_{\gamma_0};\gamma)=\{y_{ss,s,uu}=0\}.$$

If $DX(q_{\gamma_0};\gamma_0)$ has a real weak stable eigenvalue, we can multiply $X$ by a smooth positive function to obtain

$$B^s(y_c;\gamma)=v^s.$$

If $DX(q_{\gamma_0}; \gamma_0)$ has complex conjugate weak stable eigenvalues, we can bring $DX^c|_{\{y_{ss,\,s,\,uu}=0\}}$ into Jordan normal form by an $y_c$ dependent coordinate change. By furthermore multiplying $X$ by a positive function, we may then assume

$$B^s(y_c; \gamma) = \begin{pmatrix} \nu^s & -\omega^s(y_c; \gamma) \\ \omega^s(y_c; \gamma) & \nu^s \end{pmatrix}. \tag{3.34}$$

Here $\omega^s(0; \gamma_0) = \omega^s$, the imaginary part of the weak stable eigenvalue of $DX(q_{\gamma_0}; \gamma_0)$.

Write

$$U^c(y_c; \gamma) = (\mu(\gamma) + a(\gamma)\, y_c^2 + \mathcal{O}(y_c^3))\, \frac{\partial}{\partial y_c}, \tag{3.35}$$

for smooth functions $\mu$ and $a$ of $\gamma$. We assume that

$$a(\gamma_0) \neq 0.$$

For definiteness we will assume $a(\gamma_0) > 0$. Then $X$ has no singularities near $q_{\gamma_0}$ if $\mu(\gamma) > 0$.

Let $E^{ss} \times E^s \times E^c \times E^{uu}$ be the splitting of $\mathbb{R}^n$, corresponding to the coordinates $(y_{ss}, y_s, y_c, y_{uu})$. To keep the notation readable, we write e.g. $E^{ss,\,s,\,c}$ for $E^{ss} \times E^s \times E^c$.

The proposition, apart from the statements on $\Omega(\gamma)$, follows from Propositions A.2, A.3 and A.4 in the appendix. Note that, by Proposition A.4, the transition time $\hat{\tau}$ from $S^{in} \to S^{out}$ is a smooth function of $(y^{in}_{ss,\,s}, y^{out}_{uu}; \gamma)$. The vanishing of $\psi^s$, $U^{ss,\,s}$ and $U^{uu}$ along certain manifolds is a direct consequence of the coordinate changes.

Suppose that $DX(q_{\gamma_0}; \gamma_0)$ has complex conjugate weak stable eigenvalues. Let $y^0_c(t)$, $0 \leqslant t \leqslant \hat{\tau}(0, 0; \gamma)$ be the orbit with $y^0_c(0) = W^c(q_{\gamma_0}; \gamma) \cap S^{in}$ and $y^0_c(\hat{\tau}(0, 0; \gamma)) = W^c(q_{\gamma_0}; \gamma) \cap S^{out}$. We have

$$\Omega(\gamma) = \int_0^{\hat{\tau}(0,\,0;\,\gamma)} \omega^s(y^0_c(v); \gamma)\, dv.$$

Note that

$$\Omega(\gamma) = \int_{-1}^{1} \frac{\omega^s(w; \gamma)}{U^c(w; \gamma)}\, dw. \tag{3.36}$$

Because $\hat{\tau}(0, 0; \gamma) \sim 1/\sqrt{\mu}$, where the symbol $\sim$ means that the quotient is bounded and bounded away from zero, one estimates from (3.36) that $\Omega \sim 1/\sqrt{\mu}$ and $\|(\partial/\partial\gamma)\, \Omega\| \sim 1/\mu\sqrt{\mu}$ (compare Proposition A.4 and [12]). ∎

### 3.3. *The Poincaré Return Map in Shil'nikov Variables*

In the previous two subsections we provided asymptotic expressions for the transition maps near the hyperbolic singularity and near the saddle-node, in Shil'nikov variables. These can be combined to obtain asymptotic expressions for the Poincaré return map on $\Sigma^{in}$.

Observe that different coordinate systems were used to obtain asymptotic expansions for the two transition maps. Indeed, we have two coordinate systems $(x_{ss}^{out}, x_s^{out}, x_{uu}^{out})$ and $(y_{ss}^{in}, y_s^{in}, y_{uu}^{in})$ on the cross section $\Sigma^{out} = S^{in}$. And we have two coordinate systems $(x_{ss}^{in}, x_u^{in}, x_{uu}^{in})$ and $(y_{ss}^{out}, y_s^{out}, y_{uu}^{out})$ on $\Sigma^{in} = S^{out}$. To go from one coordinate system to another, goes by a diffeomorphic coordinate change. This step becomes trivial with the following lemma which provides convenient coordinate systems $x$ near the hyperbolic singularity and $y$ near the saddle-node.

LEMMA 3.3. *There are coordinate systems $x$ satisfying* (3.8), (3.9), (3.11), (3.10) *and $y$ satisfying* (3.24), (3.25), (3.26), (3.27), (3.28), (3.34), *so that the following holds. If $DX(q_{\gamma_0}, \gamma_0)$ possesses a real weak stable eigenvalue, then for a parameter dependent function $\varepsilon$,*

$$x_{ss}^{in} = y_{ss}^{out}, \qquad \varepsilon + x_u^{in} = y_s^{out}, \qquad x_{uu}^{in} = y_{uu}^{out},$$

$$x_{ss}^{out} = (\pm I)^{p^{ss}}\, y_{ss}^{in}, \qquad x_s^{out} = \pm y_s^{in}, \qquad x_{uu}^{out} = (\pm I)^{p^{uu}}\, y_{uu}^{in}.$$

*If $DX(q_{\gamma_0}, \gamma_0)$ possesses complex conjugate weak stable eigenvalues, then, up to a parameter dependent function $\varepsilon$,*

$$(x_{ss}^{in}, \varepsilon + x_u^{in}) = (y_{ss}^{out}, y_s^{out}), \qquad x_{uu}^{in} = y_{uu}^{out},$$

$$(x_{ss}^{out}, x_s^{out}) = (\pm I)^{p^{ss}+2}\,(y_{ss}^{in}, y_s^{in}), \qquad x_{uu}^{out} = (\pm I)^{p^{uu}}\, y_{uu}^{in}.$$

*In fact, writing $y_s = (y_{s,1}, y_{s,2})$, we have $x_s^{out} = y_{s,1}^{in}$ and $\varepsilon + x_u^{in} = y_{s,1}^{out}$.*

*Proof.* We investigate the conditions on the coordinate systems $x$ and $y$, restricted to $\Sigma^{in}$. First recall that center stable manifolds near a saddle-node are not unique. The amount of nonuniqueness can be precised as follows. Let $V$ be a $n - q^{uu}$ dimensional manifold in $S^{in}$, transverse to $W^{\mathrm{unst}}(q_{\gamma_0}) \cap S^{in}$. Then there exists a center stable manifold $W^{ss,s,c}(q_{\gamma_0}; \gamma)$ with $W^{ss,s,c}(q_{\gamma_0}; \gamma) \cap S^{in} = V$. Thus prescribing the center stable manifold at $S^{in}$, $W^{ss,s,c}(q_{\gamma_0}; \gamma)$ is uniquely defined near the saddle-node, for $\mu(\gamma) \geqslant 0$. A similar statement holds for strong stable foliations. If $\mathfrak{H}^{ss}$ is any smooth foliation with $q^{ss}$ dimensional leaves of $\Sigma^{in}$ with $\mathfrak{H}^{ss}$ transverse to $W^{s,c,uu}(q_{\gamma_0}; \gamma)$ along $W^{c,uu}(q_{\gamma_0}; \gamma) \cap \Sigma^{in}$, then $\mathfrak{H}^{ss}$ extends to an invariant strong stable foliation $\mathfrak{F}^{ss}$ near the saddle-node.

Suppose that $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue. Write $W_R^{u,uu}(p_\gamma)$ for the component of $W^{u,uu}(p_\gamma) \backslash W^{uu}(p_\gamma)$ that contains $\Gamma_1$ if

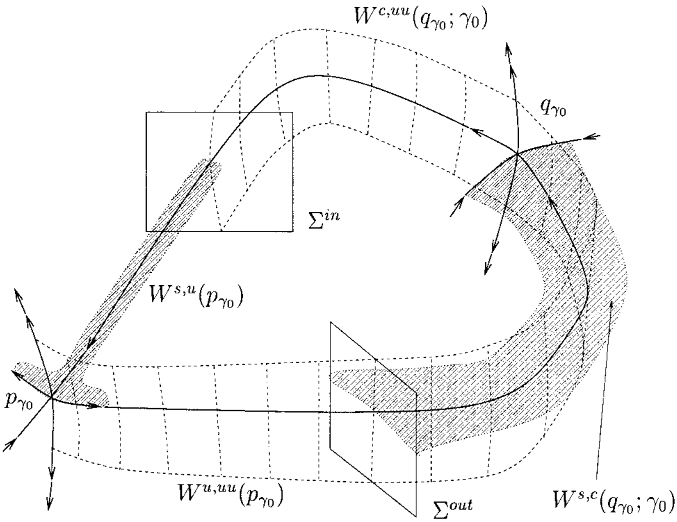



**FIG. 4.** This figure illustrates the choice of coordinates on $\Sigma^{in}$ and $\Sigma^{out}$, at $\gamma=\gamma_0$, for a three dimensional vector field with a singular cycle with a real stable and a real unstable eigenvalue at the saddle-node.

$\gamma=\gamma_0$. Similarly, write $W_R^{ss,s}(p_\gamma)$ for the component of $W^{ss,s}(p_\gamma)\backslash W^{ss}(p_\gamma)$ that contains $\Gamma_2$ if $\gamma=\gamma_0$. From the above described freedom in choosing invariant manifolds and foliations near the saddle-node, it follows that, for $\gamma=\gamma_0$, we can assume

$$W_R^{u,uu}(p_\gamma)\subset W^{c,uu}(q_{\gamma_0};\gamma_0),$$
$$W_R^{ss,s}(p_\gamma)\subset W^{ss,s,c}(q_{\gamma_0};\gamma_0).$$

For $\gamma\neq\gamma_0$ with $\mu(\gamma)>0$, we can assume

$$W_R^{u,uu}(p_\gamma)=W^{c,uu}(q_{\gamma_0};\gamma),$$
$$W_R^{ss,s}(p_\gamma)=W^{ss,s,c}(q_{\gamma_0};\gamma).$$

Moreover, for all $\gamma$ with $\mu(\gamma)\geqslant 0$, we can assume that $\mathfrak{F}^{uu}=\mathfrak{G}^{uu}$ on $W_R^{u,uu}(p_\gamma)$ and that $\mathfrak{F}^{ss}=\mathfrak{G}^{ss}$ on $W_R^{ss,s}(p_\gamma)$. In particular, $W^{ss,s}(p_\gamma)\cap\Sigma^{in}$ is a leaf of $\mathfrak{F}^{ss}$ and $W^{u,uu}(p_\gamma)\cap\Sigma^{out}$ is a leaf of $\mathfrak{F}^{uu}$. Similarly, for $\gamma\neq\gamma_0$ with $\mu(\gamma)>0$, we can assume

$$T_{W^{ss,s}(p_\gamma)}W^{ss,s,c}(q_{\gamma_0};\gamma)=T_{W^{ss,s}(p_\gamma)}W^{ss,s,u}(p_\gamma),$$
$$T_{W^{u,uu}(p_\gamma)}W^{s,c,uu}(q_{\gamma_0};\gamma)=T_{W^{u,uu}(p_\gamma)}W^{s,u,uu}(p_\gamma).$$

Observe that, by (3.8), $\{x^{in}_{u,\,uu}=0\}$ is a leaf of $\mathfrak{F}^{ss}$. Therefore, the only conditions on the coordinate systems $(x^{in}_{ss}, x^{in}_{s}, x^{in}_{uu})$ and $(y^{out}_{ss}, y^{out}_{s}, y^{out}_{uu})$ on $\Sigma^{in}$ are

$$W^{ss,\,s}(p_\gamma)\cap\Sigma^{in}=\{x^{in}_{u,\,uu}=0\},$$

$$TW^{ss,\,s,\,u}(p_\gamma)\cap\Sigma^{in}=T\{x^{in}_{uu}=0\},$$

$$TW^{s,\,c,\,uu}(q_{\gamma_0};\gamma)\cap\Sigma^{in}=T\{y^{out}_{ss}=0\},$$

$$W^{c,\,uu}(q_{\gamma_0};\gamma)\cap\Sigma^{in}=\{y^{out}_{ss,\,s}=0\}.$$

Similar conditions hold for the coordinates restricted to $\Sigma^{out}$. Note that $\{x^{out}_{ss,\,s}=0\}$ is a leaf of $\mathfrak{F}^{uu}$. The remaining conditions on the coordinate systems $(x^{out}_{ss}, x^{out}_{s}, x^{out}_{uu})$ and $(y^{in}_{ss}, y^{in}_{s}, y^{in}_{uu})$ on $\Sigma^{out}$ are

$$W^{u,\,uu}(p_\gamma)\cap\Sigma^{out}=\{x^{out}_{ss,\,s}=0\},$$

$$TW^{s,\,u,\,uu}(p_\gamma)\cap\Sigma^{out}=T\{x^{out}_{ss}=0\},$$

$$W^{ss,\,s,\,c}(q_{\gamma_0};\gamma)\cap\Sigma^{out}=\{y^{in}_{uu}=0\}.$$

The lemma follows from the transversality conditions (TR). A similar reasoning applies if $DX(q_{\gamma_0};\gamma_0)$ has complex conjugate weak stable eigenvalues. ∎

Using coordinate systems as given by the above lemma, Proposition 3.2 provides exponential expansions for the global transition map $\Pi_{far}=\Phi_{loc}\colon\Sigma^{out}\to\Sigma^{in}$,

$$\Pi_{far}(x^{out}_{ss}, x^{out}_{s}, x^{out}_{uu};\gamma)=(x^{in}_{ss}, x^{in}_{u}, x^{in}_{uu}),$$

defined for $\mu(\gamma)>0$. Combining these with exponential expansions for the local transition map $\Pi_{loc}\colon\Sigma^{in}\to\Sigma^{out}$ from Proposition 3.1, one obtains exponential expansions for the Poincaré return map $\Pi=\Pi_{far}\circ\Pi_{loc}$. Recall that $\hat{\tau}(0;\gamma)$ is the passage time of the orbit from $\Sigma^{out}$ to $\Sigma^{in}$ that forms the center manifold $W^{c}(q_{\gamma_0};\gamma)$, and that $\rho=e^{-\hat{\tau}(0;\gamma)}$ (see Proposition A.4).

PROPOSITION 3.4. *Take coordinates $(x^{in}_{ss}, x^{in}_{u}, x^{in}_{uu})$ on $\Sigma^{in}$ as in Lemma 3.3. Let $\Pi$ be the Poincaré return map on $\Sigma^{in}$,*

$$\Pi(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j};\gamma)=(x^{in}_{ss,\,j+1}, x^{in}_{u,\,j+1}, x^{in}_{uu,\,j+1}). \tag{3.57}$$

*Then* $(x^{in}_{ss,\,j+1}, x^{in}_{u,\,j+1}, x^{in}_{uu,\,j})$ *can be written as functions of* $(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1})$:

$$\begin{aligned} x^{in}_{ss,\,j+1} &= G^{ss}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma),\\ x^{in}_{u,\,j+1} &= G^{u}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma),\\ x^{in}_{uu,\,j} &= G^{uu}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma), \end{aligned}$$

*where*

$$\begin{aligned} G^{ss} &= \rho^{\nu^s} U^{ss}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma),\\ G^{u} &= \varepsilon(\gamma) + \rho^{\nu^s} h(\gamma)(x^{in}_{u,\,j})^{\beta}\, \psi^s(x^{in}_{ss,\,j}, x^{in}_{uu,\,j+1}; \gamma) +\\ &\quad \rho^{\nu^s} h(\gamma)\, U^{u}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma),\\ G^{uu} &= U^{uu}(x^{in}_{ss,\,j}, x^{in}_{u,\,j}, x^{in}_{uu,\,j+1}; \gamma). \end{aligned}$$

*The function* $(x^{in}_{ss,\,j}, x^{in}_{uu,\,j+1}; \gamma) \mapsto \psi^s(x^{in}_{ss,\,j}, x^{in}_{uu,\,j+1}; \gamma)$ *is smooth. The functions* $\gamma \mapsto \varepsilon(\gamma)$, $h(\gamma)$, $\rho(\gamma)$ *are smooth*; $\rho$ *and its derivatives are flat functions as* $\mu(\gamma) \to 0$. *The maps* $U^{ss,}$, $U^{u}$, $U^{uu}$ *are smooth for* $x^{in}_{u,\,j} > 0$. *For some* $\omega > 0$, $\sigma > 0$ *and constants* $C_{k+l} > 0$,

$$\left\| D^k \frac{\partial^l}{\partial (x^{in}_{u,\,j})^l} U^{ss}(x^{in}_{ss,\,u,\,j}, x^{in}_{uu,\,j+1}; \gamma) \right\| \leqslant C_{k+l}(x^{in}_{u,\,j})^{\beta+\omega-l},$$

$$\left\| D^k \frac{\partial^l}{\partial (x^{in}_{u,\,j})^l} U^{u}(x^{in}_{ss,\,u,\,j}, x^{in}_{uu,\,j+1}; \gamma) \right\| \leqslant C_{k+l}(x^{in}_{u,\,j})^{\beta+\omega-l},$$

$$\left\| D^k \frac{\partial^l}{\partial (x^{in}_{u,\,j})^l} U^{uu}(x^{in}_{ss,\,u,\,j}, x^{in}_{uu,\,j+1}; \gamma) \right\| \leqslant C_{k+l}(x^{in}_{u,\,j})^{\bar{\lambda}^{uu}/\lambda^{u}}.$$

*Here* $D^k$ *stands for* $k^{\text{th}}$ *order derivative with respect to* $(x^{in}_{ss,\,j}, x^{in}_{uu,\,j+1}, \gamma)$.

*The function* $h$ *satisfies the following properties*:

- *If* $DX(q_{\gamma_0}; \gamma_0)$ *has a real weak stable eigenvalue, and* $\mathbf{F}^{ss} \oplus \mathbf{F}^{uu}$, *defined by* (2.4), (2.3), *forms an orientable bundle along* $\overline{\Gamma_1 \cup \Gamma_2}$, *then* $h > 0$ *for* $\gamma$ *near* $\gamma_0$.
- *If* $DX(q_{\gamma_0}; \gamma_0)$ *has a real weak stable eigenvalue, and* $\mathbf{F}^{ss} \oplus \mathbf{F}^{uu}$ *forms a nonorientable bundle along* $\overline{\Gamma_1 \cup \Gamma_2}$, *then* $h < 0$ *for* $\gamma$ *near* $\gamma_0$.
- *If* $DX(q_{\gamma_0}; \gamma_0)$ *has complex conjugate weak stable eigenvalues, then*

$$h(\gamma) = F_1(\gamma)\cos(\Omega(\gamma)),$$

*for some smooth function* $F_1 \neq 0$. *The function* $\Omega/\tau$ *is bounded and bounded away from* 0, $\|(\partial/\partial\gamma)\,\Omega\|$ *goes to* $\infty$ *as* $\mu \to 0$.

*Remark.* In case $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue, we actually have $G^{ss} = \mathcal{O}(\rho^{\bar{\nu}^{ss}} (x^{in}_{u,j})^{\beta+\omega})$.

*Proof.* By Proposition 3.1 we can write, for some $\omega > 0$,

$$x^{out}_{ss,j} = \mathcal{O}((x^{in}_{u,j})^{-\bar{\lambda}^{ss}/\lambda^u}), \tag{3.37}$$

$$x^{out}_{s,j} = (x^{in}_{u,j})^{\beta} \phi^s(x^{in}_{ss,j}, x^{out}_{uu,j}; \gamma) + \mathcal{O}((x^{in}_{u,j})^{\beta+\omega}), \tag{3.38}$$

$$x^{in}_{uu,j} = \mathcal{O}((x^{in}_{u,j})^{\bar{\lambda}^{uu}/\lambda^u}). \tag{3.39}$$

By Proposition 3.2 and Lemma 3.3, we can write, for some $\sigma > 0$

$$x^{in}_{ss,j+1} = \rho^{\nu^s} \mathcal{O}(\| x^{out}_{ss,s,j} \|), \tag{3.40}$$

$$\begin{aligned} x^{in}_{u,j+1} &= \varepsilon(\gamma) + \rho^{\nu^s} h(\gamma)\, \psi^s(x^{out}_{ss,s,j}, x^{in}_{uu,j+1}; \gamma) \\ &\quad + \rho^{\nu^s+\sigma} h(\gamma)\, \mathcal{O}(\| x^{out}_{ss,s,j} \|), \end{aligned} \tag{3.41}$$

$$x^{out}_{uu,j} = \rho^{\bar{\nu}^{uu}} \mathcal{O}(\| x^{in}_{uu,j+1} \|). \tag{3.42}$$

Here $h$ is as in the statement of the proposition; its properties are clear from Lemma 3.3.

From equations (3.37), (3.38) and (3.42), we can solve $(x^{out}_{ss,j}, x^{out}_{s,j}, x^{out}_{uu,j})$ as functions of $(x^{in}_{ss,j}, x^{in}_{u,j}, x^{in}_{uu,j+1})$ by the implicit function theorem, for $\rho$ and $x^{in}_{u,j}$ small. Putting this in the remaining equations proves the proposition. ∎

## 4. BIFURCATION EQUATIONS

In this section we derive bifurcation equations for $N$-homoclinic orbits and $N$-periodic orbits. These equations will then be solved for $N = 1, 2$. The absence of $N$-periodic orbits and $N$-homoclinic orbits for $N \geqslant 3$ will be established by using geometric methods such as the construction of invariant manifolds and foliations, in the next section.

Let $x^{in}_{j+1} = \Pi(x^{in}_j; \gamma)$ be an orbit of the Poincaré return map $\Pi\colon \Sigma^{in} \to \Sigma^{in}$. For an $N$-periodic orbit, $x^{in}_N = x^{in}_0$ and $x^{in}_{u,j} > 0$ for all $j$. For an $N$-homoclinic orbit, $x^{in}_N = x^{in}_0$, $x^{in}_{u,0} = x^{in}_{u,N} = 0$ and $x^{in}_{u,j} > 0$ for $0 < j < N$.

Define the map $\Psi_j$ depending on $(x^{in}_{ss,j+1}, x^{in}_{u,j+1}, x^{out}_{uu,j})$ by

$$\Psi_j = \begin{pmatrix} x^{in}_{ss,j+1} \\ x^{in}_{uu,j+1} \\ x^{in}_{uu,j} \end{pmatrix} - \begin{pmatrix} G^{ss}(x^{in}_{ss,j}, x^{in}_{u,j}, x^{in}_{uu,j+1}; \gamma) \\ G^{u}(x^{in}_{ss,j}, x^{in}_{u,j}, x^{in}_{uu,j+1}; \gamma) \\ G^{uu}(x^{in}_{ss,j}, x^{in}_{u,j}, x^{in}_{uu,j+1}; \gamma) \end{pmatrix},$$

where $G^{ss}$, $G^u$, $G^{uu}$ are given by Proposition 3.6.

Write $\mathbf{\Psi} = (\Psi_0, ..., \Psi_{N-1})$ and likewise $\mathbf{x}_{ss}^{in} = (x_{ss,0}^{in}, ..., x_{ss,N-1}^{in})$, etc. Note that $\mathbf{\Psi} = \mathbf{0}$ at $N$-periodic and $N$-homoclinic orbits. Let $\mathbf{P}$ be the orthogonal projection onto the image $\operatorname{Im} D_{(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{in})} \mathbf{\Psi}|_{\mathbf{x}_u^{in} = \mathbf{0}}$. Performing a Lyapunov–Schmidt reduction we split the equation $\mathbf{\Psi} = \mathbf{0}$ into the equations $(\mathbf{I} - \mathbf{P})\mathbf{\Psi} = \mathbf{0}$ and $\mathbf{P}\mathbf{\Psi} = \mathbf{0}$. We then solve $(\mathbf{I} - \mathbf{P})\mathbf{\Psi} = \mathbf{0}$ for $(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{in})$ as functions of $\mathbf{x}_u^{in}$ and, putting this into $\mathbf{P}\mathbf{\Psi} = \mathbf{0}$, obtain reduced bifurcation equations. The following proposition treats this reduction.

PROPOSITION 4.1. *The equation $(\mathbf{I} - \mathbf{P})\mathbf{\Psi} = \mathbf{0}$ can be solved for $(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{in})$ as functions of $\mathbf{x}_u^{in}$ and $\gamma$. Putting these solutions into the equation $\mathbf{P}\mathbf{\Psi} = \mathbf{0}$ one obtains the reduced bifurcation equation*

$$x_{u,j+1}^{in} = \varepsilon(\gamma) + \rho^{\nu^s} h(\gamma)(x_{u,j}^{in})^{\beta} + \rho^{\nu^s} h(\gamma)\, U(\mathbf{x}_u^{in}; \gamma). \tag{4.43}$$

*Here $\varepsilon$ and $h$ are smooth functions of $\gamma$; $h$ satisfies the same properties as in Proposition 3.6. For the higher order term $U(\mathbf{x}_u^{in}; \gamma)$, the following estimates hold*:

$$\frac{\partial^{k+l}}{\partial(\mathbf{x}_u^{in})^k\, \partial\gamma^l} U(\mathbf{x}_u^{in}; \gamma) = \mathcal{O}(\|\mathbf{x}_u^{in}\|^{\beta+\omega-k}),$$

*for some $\omega > 0$, $\sigma > 0$.*

*Proof.* Compute

$$D_{x_{ss,j+1}^{in}} \Psi_j|_{\mathbf{x}_u^{in} = \mathbf{0}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.44}$$

and

$$D_{x_{uu,j}^{out}} \Psi_j|_{\mathbf{x}_u^{in} = \mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.45}$$

We conclude that $D_{(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{out})} \mathbf{\Psi}$ has maximal rank $N(n-1)$. Hence, we can solve $(\mathbf{I} - \mathbf{P})\mathbf{\Psi} = \mathbf{0}$ by the implicit mapping theorem and obtain $(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{out})$ as functions of $\mathbf{x}_u^{in}$ and $\gamma$. Observe that $\operatorname{Im} D_{(\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{out})} \mathbf{\Psi}|_{\mathbf{x}_u^{in} = \mathbf{0}}$ is independent of $\gamma$ and $\mathbf{x}_{ss}^{in}, \mathbf{x}_{uu}^{out}$.

Note that, if $\mathbf{x}_u^{in}=\mathbf{0}$, then $\mathbf{x}_{ss}(\mathbf{0};\gamma)=(0,...,0)$ and $\mathbf{x}_{uu}^{out}(\mathbf{0};\gamma)=(0,...,0)$. It follows from the implicit mapping theorem and the estimates on $\mathbf{\Psi}$ that

$$\mathbf{x}_{ss}^{in}(\mathbf{x}_u^{in};\gamma)=\rho^{\nu^s}\mathcal{O}(\|\mathbf{x}_u^{in}\|^{\beta}), \tag{4.46}$$

$$\mathbf{x}_{uu}^{in}(\mathbf{x}_u^{in};\gamma)=\mathcal{O}(\|\mathbf{x}_u^{in}\|^{\bar{\lambda}^{uu}/\lambda^u}). \tag{4.47}$$

Similar statements hold for higher order derivatives. ▮

### 4.1. *Proofs of the Bifurcation Theorems*

Before computing the bifurcation curves in the different bifurcation theorems, we indicate the reparametrization of the parameter plane. Let $\mu$ be defined by (3.35) and $\varepsilon$ by Proposition 4.1. Of course, this only defines $\varepsilon$ if $\mu>0$. Note that $\varepsilon$ equals the $x_u$ coordinate of the first intersection of $W^u(p_\gamma)$ with $\Sigma^{in}$. By defining $\varepsilon$, for $\mu\leqslant 0$, as the $x_u$ coordinate of the first intersection of $W^c(q_{\gamma_0};\gamma)$ with $\Sigma^{in}$, it is clear that $\varepsilon$ is a smooth function of $\gamma$, for all $\gamma$ near $\gamma_0$. Summarizing,

$$\varepsilon=\begin{cases}\pi_u(W^u(p_\gamma)\cap\Sigma^{in}), & \mu>0,\\ \pi_u(W^c(q_{\gamma_0};\gamma)\cap\Sigma^{in}), & \mu\leqslant 0,\end{cases} \tag{4.48}$$

where $\pi_u$ is the coordinate projection onto $E^u$.

By the generic unfolding condition (GU), we may assume that

$$\gamma=(\mu,\varepsilon).$$

*Proof of Theorem* 2.1. This theorem follows easily once noticed that the Poincaré return map $\Pi$ restricted to the intersection of the center stable manifold $W^{ss,s,u}(\overline{\Gamma_1\cup\Gamma_2};\gamma)$ (see Proposition 5.1) with the cross section $\Sigma^{in}$, is a contraction. We leave details to the reader. ▮

*Proof of Theorem* 2.2. In the reduced bifurcation equation (4.43) from Proposition 4.1, we have

$$h(\gamma)>0,$$

for $\gamma$ near $\gamma_0$. For 1-periodic orbits, the reduced bifurcation equation is of the form

$$x_u=\varepsilon+h\rho^{\nu^s}x_u^{\beta}+\rho^{\nu^s}\mathcal{O}(x_u^{\beta+\omega}). \tag{4.49}$$

To solve for saddle-node bifurcations of 1-periodic orbits, we obtain an additional equation by differentiating (4.49),

$$1=h\beta\rho^{\nu^s}x_u^{\beta-1}+\rho^{\nu^s}\mathcal{O}(x_u^{\beta+\omega-1}). \tag{4.50}$$

Solving this equation for $x_u$ by the implicit function theorem we get

$$x_u = (h\beta\rho^{\nu^s})^{1/(1-\beta)} + o((\rho^{\nu^s})^{1/(1-\beta)}). \tag{4.51}$$

Equation (4.49) can be solved for $\varepsilon$ as function of $\mu$ and $x$, applying the implicit function theorem. With (4.1) this yields

$$\varepsilon = -\frac{1-\beta}{\beta}\,(h(0)\,\beta\rho^{\nu^s})^{1/(1-\beta)} + o((\rho^{\nu^s})^{1/(1-\beta)})$$

We claim that 2-periodic orbits and 2-homoclinic orbits do not exist. For this, one writes down a bifurcation equation as (4.52) below in the proof of Theorem 2.3. It is easily seen that this does not have a solution if $h(\gamma) > 0$. By Proposition 5.1 below, $N$-periodic orbits or $N$-homoclinic orbits for $N > 2$ can not exist.

It remains to prove the statements on the dimensions of the stable and unstable manifolds of periodic orbits. These are however immediate consequences of the bifurcation results plus Proposition 5.1. ∎

*Proof of Theorem* 2.3. In the reduced bifurcation equation (4.43) from Proposition 4.1, we have

$$h(\gamma) < 0,$$

for $\gamma$ near $\gamma_0$. We will restrict attention to 2-periodic and 2-homoclinic orbits; 1-periodic and 1-homoclinic orbits can be treated as above.

2-*homoclinic orbits*. For a two-homoclinic orbit, the bifurcation equations are of the form

$$\begin{aligned} x_{u,1} &= \varepsilon + h\rho^{\nu^s} x_{u,0}^{\beta} + U_1(x_{u,0}, x_{u,1}; \gamma), \\ x_{u,0} &= \varepsilon + h\rho^{\nu^s} x_{u,1}^{\beta} + U_0(x_{u,0}, x_{u,1}; \gamma), \end{aligned}$$

with $x_{u,0} = 0$ and $x_{u,1} > 0$. Thus we get

$$\begin{aligned} x_{u,1} &= \varepsilon, \\ 0 &= \varepsilon + h\rho^{\nu^s} x_{u,1}^{\beta} + U_0(0, x_{u,1}; \gamma). \end{aligned}$$

Solving this one obtains

$$\varepsilon = (-h(0)\,\rho^{\nu^s})^{1/(1-\beta)} + o((\rho^{\nu^s})^{1/(1-\beta)}).$$

2-*periodic orbits*. For a 2-periodic orbit, the bifurcation equations are of the form

$$x_{u,1} = \varepsilon + h\rho^{\nu^s} x_{u,0}^{\beta} + U_1(x_{u,0}, x_{u,1}; \gamma),$$

$$x_{u,0} = \varepsilon + h\rho^{\nu^s} x_{u,1}^{\beta} + U_0(x_{u,0}, x_{u,1}; \gamma),$$

with $x_{u,0}$ and $x_{u,1}$ positive. We may assume that $x_{u,0} < x_{u,1}$ and write $x_{u,1} = x$, $x_{u,0} = ax$ for some $0 < a < 1$. The equations to solve are then

$$x = \varepsilon + h\rho^{\nu^s} a^{\beta} x^{\beta} + U_1(ax, x; \gamma),$$

$$ax = \varepsilon + h\rho^{\nu^s} x^{\beta} + U_0(ax, x; \gamma).$$

By symmetry, $U_0(ax, x; \gamma) = U_1(x, ax; \gamma)$. Subtracting the both equations and dividing by $(1-a)\, x^{\beta}$ yields

$$x^{1-\beta} = -h\rho^{\nu^s} \frac{1-a^{\beta}}{1-a} + \frac{U_1(ax, x; \gamma) - U_1(x, ax; \gamma)}{(1-a)\, x^{\beta}}. \tag{4.52}$$

Here $U_1(ax, x; \gamma) - U_1(x, ax; \gamma) = \rho^{\nu^s} \mathcal{O}((1-a)\, x^{\beta+\omega})$ as $a \to 1$. It follows that (4.52) has a well defined limit as $a \to 1$,

$$x^{1-\beta} = -h\rho^{\nu^s} \beta + \bar{U}(x; \gamma).$$

Here $\bar{U}(x; \gamma) = \rho^{\nu^s} \mathcal{O}(x^{\omega})$. We thus obtain period-doubling bifurcations if $\rho^{\nu^s} < 0$ and

$$\varepsilon = (\beta^{1/(1-\beta)} + \beta^{\beta/(1-\beta)})(-h(0)\, \rho^{\nu^s})^{1/(1-\beta)} + o((\rho^{\nu^s})^{1/(1-\beta)}).$$

A straightforward computation shows that the period-doubling bifurcation is supercritical.

By Proposition 5.1, below, $N$-periodic orbits or $N$-homoclinic orbits for $N > 2$ can not exist. The statements on the dimensions of the stable and unstable manifolds of periodic orbits follow from the bifurcation results plus Proposition 5.1. ∎

*Proof of Theorem* 2.4. We first solve for bifurcations of $n$-periodic and $n$-homoclinic orbits, $n = 1, 2$. The analysis is similar to the computations used to prove Theorems 2.2, 2.3, with some adaptations to cope with the fact that $h$ does not have a definite sign. We treat bifurcations of 1-periodic orbits; bifurcations of 2-homoclinic orbits and 2-periodic orbits can be analyzed analogously, with similar adaptations.

1-*periodic orbits.* As in the proof of Theorem 2.2 one obtains a bifurcation equation for saddle-node bifurcations of 1-periodic orbits;

$$\varepsilon = -\frac{1-\beta}{\beta}\,(h(\gamma)\,\beta\rho^{\nu^s})^{1/(1-\beta)} + o((\rho^{\nu^s})^{1/(1-\beta)})$$

with $h > 0$. To see what this gives in the $(\varepsilon, \mu)$-parameter plane, one must know how $(\varepsilon, \mu) \mapsto (\varepsilon, h(\varepsilon, \mu))$ folds the parameter plane.

LEMMA 4.2. *$h$ vanishes along a countable set of smooth curves $\{(\varepsilon, \pi_j(\varepsilon))\}$, satisfying $\pi_j > 0$ for all $j$ and* $\lim_{j\to\infty} \pi_j = 0$. *Furthermore,*

$$\frac{\partial}{\partial\mu}\, h|_{(\varepsilon,\, \pi_j(\varepsilon))} \neq 0,$$

*for all small $\varepsilon$ and $j > 0$.*

*Proof.* Compute

$$\frac{\partial}{\partial\mu}\, h = -F_1 \sin(\Omega)\, \frac{\partial}{\partial\mu}\, \Omega + \cos(\Omega)\, \frac{\partial}{\partial\mu}\, F_1\,.$$

Assume $h = 0$. Since $(\partial/\partial\mu)\, \Omega$ gets arbitrarily large as $\mu \to 0$, the first term on the righthandside of (4.53) is the most important. It follows that $h$ and $(\partial/\partial\mu)\, h$ can not both vanish at a same parameter value. The lemma now follows from elementary considerations. ∎

The statement on the curves of saddle-node bifurcations of 1-periodic orbits follows.

Remains to show the nonexistence of $n$-periodic orbits and $n$-homoclinic orbits for $n > 2$. By Proposition 5.1, a $C^{1+\varepsilon}$ locally invariant center stable manifold $W^{ss,s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma)$ exists near $\overline{\Gamma_1 \cup \Gamma_2}$, for some $\varepsilon > 0$. Restrict the Poincaré return map $\Pi$ to $\Sigma^{in} \cap W^{ss,s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma)$. The manifold $W^{ss,s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma) \cap \Sigma^{in}$ defines $x^{in}_{uu,\, j+1}$ as function of $(x^{in}_{ss,\, j+1}, x^{in}_{u,\, j+1})$. By the implicit mapping theorem, applied to the expansions in Proposition 3.4, one gets $(x^{in}_{ss,\, j+1}, x^{in}_{u,\, j+1})$ as functions of $(x^{in}_{ss,\, j}, x^{in}_{u,\, j})$. Thus one obtains the following expression for $\Pi$:

$$\Pi(x_{ss}, x_u; \gamma) = \begin{pmatrix} \rho^{\nu^s} g(\gamma)\, x_u^{\beta} \psi^s(x_{ss}; \gamma) + \rho^{\nu^s} \mathcal{O}(x_u^{\beta+\omega}) \\ \varepsilon(\gamma) + \rho^{\nu^s} h(\gamma)\, x_u^{\beta} \psi^s(x_{ss}; \gamma) + \rho^{\nu^s} h(\gamma) \mathcal{O}(x_u^{\beta+\omega}) \end{pmatrix}.$$

We will cover a neighborhood $\mathscr{W}$ of $\gamma=0$ in the parameter plane by two regions $\mathscr{W}_1$ and $\mathscr{W}_2$. Take $\mathscr{W}_1$ to consist of parameters $\gamma=(\varepsilon,\mu)$ near $(0,0)$ for which $\rho^{\nu^s}|h(\gamma)|$ divided by $\varepsilon^{1-\beta}$ is bounded, and define $\mathscr{W}_2$ as the complement. For $\gamma$ from $\mathscr{W}_1$, a strong stable foliation for $\Pi$ will be constructed. Dynamics of $\Pi$ for parameters from $\mathscr{W}_2$ is studied separately. We note that the statements on the dimensions of the stable and unstable manifolds of periodic orbits follow from these considerations, combined with Proposition 5.1.

*Parameters from* $\mathscr{W}_1$. Let $k$ be defined by $\varepsilon=k(\rho^{\nu^s}|h|)^{1/(1-\beta)}$. We consider parameters $(\varepsilon,\mu)$ for which $k$ is from a bounded interval $[-E,E]$. This defines $\varepsilon$ as function of $k$ and $\mu$, so that we can consider $\mu$ and $k$ as new parameters.

Consider the rescaling defined by

$$\begin{pmatrix} x_{ss} \\ x_u \end{pmatrix}=\begin{pmatrix} \rho^{\nu^s}|h| & g\,\operatorname{sign}(h)(\rho^{\nu^s}|h|)^{\beta/(1-\beta)} \\ 0 & (\rho^{\nu^s}|h|)^{1/(1-\beta)} \end{pmatrix}\begin{pmatrix} \bar{x}_{ss} \\ \bar{x}_u \end{pmatrix}$$

The Poincaré return map $(\bar{x}_{ss},\bar{x}_u)\mapsto\bar{\Pi}(\bar{x}_{ss},\bar{x}_u;\mu,k)$ in rescaled coordinates and with parameters $(\mu,k)$, has an expression

$$\bar{\Pi}(\bar{x}_{ss},\bar{x}_u;\mu,k)=\begin{pmatrix} -gk\,\operatorname{sign}(h)(\rho^{\nu^s}|h|)^{(2\beta-1)/(1-\beta)} \\ k+\operatorname{sign}(h)\,\bar{x}_u^{\beta}\psi^s \end{pmatrix}$$
$$+(\rho^{\nu^s}|h|)^{\omega/(1-\beta)}\,\mathcal{O}(\bar{x}_u^{\beta+\omega}),$$

where $\psi^s$ is evaluated at $(x_{ss};\gamma)=(\rho^{\nu^s}|h|\,\bar{x}_u+g\,\operatorname{sign}(h)(\rho^{\nu^s}|h|)^{\beta/(1-\beta)}\,\bar{x}_{ss};\gamma)$. As $\rho^{\nu^s}h\to 0$, $\bar{\Pi}(\bar{x}_{ss},\bar{x}_u;\mu,k)\to\bar{\Pi}_0(\bar{x}_{ss},\bar{x}_u;\mu,k)$, where

$$\bar{\Pi}_0(\bar{x}_{ss},\bar{x}_u;\mu,k)=\begin{pmatrix} 0 \\ k+\operatorname{sign}(h)\,\bar{x}_u^{\beta}\psi^s(0) \end{pmatrix}.$$

This convergence is uniform for $(\bar{x}_{ss},\bar{x}_u)\in[-I,I]\times(0,I]$, where $I$ is a positive constant. By Proposition 5.2, $\bar{\Pi}$ possesses a differentiable strong stable foliation. Therefore, $\bar{\Pi}$ does not have any $n$-periodic orbits or $n$-homoclinic orbits, for $n>2$.

*Parameters from* $\mathscr{W}_2$. The region $\mathscr{W}_2$ is the complement of $\mathscr{W}_1$ and therefore given by

$$\mathscr{W}_2=\{(\varepsilon,\mu);\ |\varepsilon|\geqslant E(\rho^{\nu^s}|h|)^{1/(1-\beta)}\}.$$

For parameters from $\mathscr{W}_2$, one shows that, if $E$ is taken large enough, $\{X(\cdot\,;\gamma)\}$ possesses just an attracting 1-periodic orbit if $\varepsilon>0$ and no periodic orbits if $\varepsilon<0$.

For $\gamma=(\varepsilon,\mu)\in\mathscr{W}_2$, let $k$ be given by $\rho^{\nu^s}h=k\,|\varepsilon|^{1-\beta}$; $k$ is contained in $[-(1/E)^{1-\beta},(1/E)^{1-\beta}]$. Consider rescaled coordinates $(\hat{x}_{ss},\hat{x}_u)$ given by

$$\begin{pmatrix} x_{ss} \\ x_u \end{pmatrix}=\begin{pmatrix} |\varepsilon|^{1-\beta} & 0 \\ 0 & |\varepsilon| \end{pmatrix}\begin{pmatrix} \hat{x}_{ss} \\ \hat{x}_u \end{pmatrix}$$

Computing the Poincaré return map $(\hat{x}_{ss},\hat{x}_u)\mapsto\hat{\Pi}(\hat{x}_{ss},\hat{x}_u;\gamma)$ in rescaled coordinates, one gets

$$\hat{\Pi}(\hat{x}_{ss},\hat{x}_u;\gamma)=\begin{pmatrix} 0 \\ \operatorname{sign}(\varepsilon)+k\hat{x}_u^{\beta}\psi^s \end{pmatrix}+\mathcal{O}(|\varepsilon|^{\tilde{\omega}}\,\hat{x}_u^{\beta}),$$

for some $\tilde{\omega}>0$. Here $\psi^s$ is evaluated at $(x_{ss};\gamma)=(|\varepsilon|^{1-\beta}\,\hat{x}_{ss};\gamma)$. When we let $\varepsilon$ go to 0, then $\hat{\Pi}(\hat{x}_{ss},\hat{x}_u;\gamma)\to\hat{\Pi}_0(\hat{x}_{ss},\hat{x}_u;\gamma)$ given by

$$\hat{\Pi}_0(\hat{x}_{ss},\hat{x}_u;\gamma)=\begin{pmatrix} 0 \\ \operatorname{sign}(\varepsilon)+k\hat{x}_u^{\beta}\psi^s(0) \end{pmatrix}.$$

This convergence is uniform on sets of the form $[-I,I]\times(0,I]$, where $I$ is a positive number. It is clear that $\hat{\Pi}_0$ has a stable fixed point if $\varepsilon>0$, attracting all points in its domain. If $\varepsilon<0$, all points of $\hat{\Pi}_0$ are eventually mapped outside the domain of $\hat{\Pi}_0$. If we consider only small values of $k$, i.e. if $E$ is chosen sufficiently large, then for $\varepsilon$ small and positive, $\hat{\Pi}$ has a stable fixed point in $[-I,I]\times(0,I]$, which attracts all points in $[-I,I]\times(0,I]$. And if $\varepsilon$ is small and negative, all points in $[-I,I]\times(0,I]$ are mapped outside the domain of $\hat{\Pi}$. ∎

## 5. INVARIANT MANIFOLDS AND FOLIATIONS

The nonexistence of $n$-periodic orbits and $n$-homoclinic orbits near a singular heteroclinic cycle is shown by geometric techniques, such as the construction of invariant center manifolds or invariant strong stable foliations. In this section we provide these results which were applied in the proofs of the bifurcation theorems.

A manifold $V$ is called locally invariant for the vector field $X$, if $X(x)$ is contained in $T_xV$ for each $x\in V$. A locally invariant manifold $V$ is called normally hyperbolic if $T\mathbb{R}^n|_V$ splits as $T\mathbb{R}^n|_V=E^s\oplus TV\oplus E^u$ for vector bundles $E^s$, $E^u$ over $V$ with the following properties. There are $C>0$, $\lambda>0$ so that for $t>0$, for $x\in V$ satisfying the property that $X_s(x)\in V$, $0\leqslant s\leqslant t$, and for $v_s\in E^s(x)$, $v_c\in T_xV$,

$$\|DX_t(x)\,v_s\|/\|DX_t(x)\,v_c\|\leqslant Ce^{-\lambda t}\,\|v_s\|/\|v_c\|,$$

and, moreover, so that for $t<0$, for $x \in V$ satisfying $X_s(x) \in V$, $t \leqslant s \leqslant 0$, and for $v_u \in E^u(x)$, $v_c \in T_x V$,

$$\|DX_t(x)\, v_u\| / \|DX_t(x)\, v_c\| \leqslant Ce^{-\lambda t}\, \|v_u\| / \|v_c\|.$$

Proposition 5.1 below provides normally hyperbolic invariant manifolds near $\overline{\Gamma_1 \cup \Gamma_2}$. Such a manifold contains the nonwandering set of $X$ restricted to a small neighborhood of $\overline{\Gamma_1 \cup \Gamma_2}$. The construction of such manifolds is very similar to the construction of invariant manifolds near a homoclinic orbit as in [16], [29] and will therefore be left out. See also [35].

If $\gamma = \gamma_0$ there exists a continuous bundle $\mathbf{F}^{ss,\,s,\,u}$ along $\overline{\Gamma_1 \cup \Gamma_2}$;

$$\mathbf{F}^{ss,\,s,\,u} = \bigcup_{x \in \Gamma_2 \cup p_{\gamma 0}} T_x W^{ss,\,s,\,u}(p_{\gamma_0}) \cup \bigcup_{y \in \Gamma_1 \cup q_{\gamma 0}} T_y W^{\mathrm{st}}(q_{\gamma_0}).$$

With $\mathbf{F}^{uu}$ given by (2.3), we have

$$T\mathbb{R}^n |_{\overline{\Gamma_1 \cup \Gamma_2}} = \mathbf{F}^{ss,\,s,\,u} \oplus \mathbf{F}^{uu}.$$

Moreover, if $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue, the bundle $\mathbf{F}^{ss,\,s,\,u}$ is a Whitney sum $\mathbf{F}^{ss} \oplus \mathbf{F}^{s,\,u}$ of two continuous bundles, where $\mathbf{F}^{ss}$ is given by (2.3) and $\mathbf{F}^{s,\,u}$ by

$$\mathbf{F}^{s,\,u} = \bigcup_{x \in \Gamma_1 \cup p_{\gamma 0}} T_x W^{s,\,u}(p_{\gamma_0}) \cup \bigcup_{y \in \Gamma_2 \cup q_{\gamma 0}} T_y W^{c,\,uu}(q_{\gamma_0}; \gamma_0).$$

Let $\mathbf{F}^{s,\,u,\,uu} = \mathbf{F}^{s,\,u} \oplus \mathbf{F}^{uu}$.

PROPOSITION 5.1. *Let $\{X(\,\cdot\,; \gamma)\}$ be a two parameter family of vector fields as in Section 2. Then, for $\gamma$ near $\gamma_0$, there exists a $(p^{ss}+2)$ dimensional normally hyperbolic, locally invariant manifold $W^{ss,\,s,\,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma)$ near $\overline{\Gamma_1 \cup \Gamma_2}$. This manifold is $C^{1+\varepsilon}$ and depends $C^{1+\varepsilon}$ on $\gamma$, for some $\varepsilon > 0$. At $\gamma = \gamma_0$,*

$$TW^{ss,\,s,\,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma_0) |_{\overline{\Gamma_1 \cup \Gamma_2}} = \mathbf{F}^{ss,\,s,\,u}.$$

*If $DX(q_{\gamma_0}; \gamma_0)$ has a real weak stable eigenvalue, there is moreover a $(p^{uu}+2)$ dimensional normally hyperbolic, locally invariant manifold $W^{s,\,u,\,uu}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma)$ near $\overline{\Gamma_1 \cup \Gamma_2}$. This manifold is $C^{1+\varepsilon}$ and depends $C^{1+\varepsilon}$ on $\gamma$, for some $\varepsilon > 0$. At $\gamma = \gamma_0$,*

$$TW^{s,\,u,\,uu}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma_0) |_{\overline{\Gamma_1 \cup \Gamma_2}} = \mathbf{F}^{s,\,u,\,uu}.$$

*The intersection*

$$W^{s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma) = W^{ss,s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma) \cap W^{s,u,uu}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma),$$

*is transversal, and thus gives a two dimensional normally hyperbolic invariant manifold, which is also* $C^{1+\varepsilon}$ *for some* $\varepsilon > 0$. *At* $\gamma = \gamma_0$,

$$TW^{s,u}(\overline{\Gamma_1 \cup \Gamma_2}; \gamma_0)|_{\overline{\Gamma_1 \cup \Gamma_2}} = \mathbf{F}^{s,u}.$$

Consider a box $I_1 \times I_2$, for open regions $I_1, I_2$ in Euclidean spaces. Let $\mathfrak{F} = \{\text{graph } f_{x_2}, x_2 \in I_2\}$, with $f_{x_2}: I_1 \to I_2$, be a foliation of $B$. Each leaf of $\mathfrak{F}$ is assumed to be smooth, i.e. for fixed $x_2 \in I_2$, the function $x_1 \mapsto f_{x_2}(x_1)$ is smooth. Recall that the foliation $\mathfrak{F}$ is called continuous if $x_2 \mapsto f_{x_2}(x_1)$ is continuous for each $x_1 \in I_1$. In other words, $\mathfrak{F}$ is continuous if a continuous coordinate change brings $\mathfrak{F}$ to the affine foliation $\{I_1 \times \{x_2\}, x_2 \in I_2\}$.

PROPOSITION 5.2. *Let* $(x_{ss}, x_u) \mapsto \Pi(x_{ss}, x_u; \gamma)$ *be a map on a compact set* $\Delta = \{\|x_{ss}\| \leqslant L, |x_u| \leqslant L\}$, *with an expression*

$$\Pi(x_{ss}, x_u; \gamma) = \begin{pmatrix} 0 \\ k + \operatorname{sign}(h)\, x_u^\beta \psi^s(x_{ss}; \gamma) \end{pmatrix} + \mathcal{O}(|h|^\zeta\, |x_u|^{\beta+\omega}),$$

*where* $\zeta > 0$. *Suppose the higher order terms can be differentiated and that the derivative is of* $\mathcal{O}(|h|^\zeta\, |x_u|^{\beta+\omega-1})$. *Then* $\Pi$ *possesses a continuous invariant strong stable foliation.*

*Proof.* For convenience we suppress the dependence on parameters from the notation. For $k$ from a compact interval and $L$ sufficiently large, $x_u \mapsto k + \operatorname{sign}(h)\, x_u^\beta \psi^s(x_{ss})$ maps $[-L, L]$ strictly into itself. Hence, the domain of $\Pi^{-1}$ is then strictly contained in $\Delta$.

The idea of the construction of the strong stable foliation is as follows. Take a foliation $\mathfrak{F}$ of $\Delta$ with leaves close to $\{x_u = \text{const.}\}$ and which contains, as a leaf, $\{x_u = 0\}$. We claim that $\mathfrak{F}^m = \Pi^{-m}(\mathfrak{F})$ is well defined and converges as $m \to \infty$ to a continuous foliation. The limit is the required strong stable foliation.

Let

$$C_a(x_{ss}, x_u) = \{(u, v) \in T_{(x_{ss}, x_u)}\Delta;\ |u| \leqslant a\, |v|\},$$

where $(u, v)$ are the natural coordinates on $T_{(x_{ss}, x_u)}\Delta$. Below we show that, for $\varepsilon$ small, a function $a(x_{ss}, x_u)$ exists with $0 < a(x_{ss}, x_u) \leqslant 1$ and $a(x_{ss}, x_u) \to 0$ as $x_u \to 0$, so that

$$D\Pi^{-1}(\Pi(x_{ss}, x_u))\, C_1(\Pi(x_{ss}, x_u)) \subset C_{a(x_{ss}, x_u)}(x_{ss}, x_u). \tag{5.53}$$

This means that the cone field $\{C_{a(x_{ss}, x_u)}\}$ is invariant under $D\Pi^{-1}$.

Choose the trial foliation $\mathfrak{F}$ so that $T_{(x_{ss}, x_u)} \mathfrak{F}_{(x_{ss}, x_u)} \subset C_1(x_{ss}, x_u)$. Then (5.53) implies that $\mathfrak{F}^m$ is a continuous foliation. In order to show that $\mathfrak{F}^m$ converges to a continuous foliation as $m \to \infty$, it suffices to show that for each $(x_{ss}, x_u) \in \Delta$, $(x_{ss}, x_u) \mapsto T_{(x_{ss}, x_u)} \mathfrak{F}^m_{(x_{ss}, x_u)}$ converges to a continuous line bundle over $\Delta$. Let $(x_{ss}, x_u, \sigma) \mapsto (\Pi^{-1}(x_{ss}, x_u), \Sigma(x_{ss}, x_u, \sigma))$ be the map, induced by $\Pi^{-1}$, on $\Delta \times \mathscr{L}(E^{ss}, E^u)$. That is, $\Sigma(x_{ss}, x_u, \sigma) = v$ with graph $v = D\Pi^{-1}(x_{ss}, x_u)$graph $\sigma$. This yields

$$\Sigma(x_{ss}, x_u, \sigma) = \frac{A(x_{ss}, x_u) + B(x_{ss}, x_u)\,\sigma}{C(x_{ss}, x_u) + D(x_{ss}, x_u)\,\sigma}, \tag{5.54}$$

where

$$D\Pi^{-1}(x_{ss}, x_u) = \begin{pmatrix} A(x_{ss}, x_u) & B(x_{ss}, x_u) \\ C(x_{ss}, x_u) & D(x_{ss}, x_u) \end{pmatrix}.$$

We will show that $\Sigma$ contracts distances in the fibers: there is $k < 1$ so that for all $(x_{ss}, x_u) \in \Delta$,

$$|\Sigma(x_{ss}, x_u, \sigma_1) - \Sigma(x_{ss}, x_u, \sigma_2)| \leqslant k\,|\sigma_1 - \sigma_2|. \tag{5.55}$$

It is standard to derive proposition 5.2 from (5.55), using (5.53) to assure that the limit foliation is continuous at $\{x_u = 0\}$: let

$$\mathfrak{S} = \{\sigma \in C^0(\Delta, \mathscr{L}(E^{ss}, E^u));\ |\sigma(x_{ss}, x_u)| \leqslant a(x_{ss}, x_u)\},$$

and define a graph transform $\Gamma$ on $\mathfrak{S}$ by

$$(\Gamma\sigma)(x_{ss}, x_u) = \Sigma(\Pi(x_{ss}, x_u), \sigma(\Pi(x_{ss}, x_u))).$$

Then $\Gamma$ maps $\mathfrak{S}$ into itself and is a contraction. The unique fixed point is the desired strong stable foliation, compare [15].

It remains to show (5.53) and (5.55). Observe that

$$D\Pi(x_{ss}, x_u) = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{sign}(h)\, x_u^{\beta-1} \psi^s(x_{ss}) \end{pmatrix} + \mathcal{O}(|h|^{\zeta}\, x_u^{\beta+\omega-1}) \tag{5.56}$$

From (5.56) it follows that

$$x_u^{1-\beta}\, D\Pi(x_{ss}, x_u) = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{sign}(h)\, \psi^s(x_{ss}) \end{pmatrix} + \mathcal{O}(|h|^{\zeta}\, x_u^{\omega})$$

We may replace $D\Pi^{-1}$ by the inverse of the above matrix, since they induce the same action on $\mathscr{L}(E^{ss}, E^u)$. Straightforward estimates show (5.53) and (5.55). ∎

# APPENDIX: EXPONENTIAL EXPANSIONS

Consider $X$ near $q_{\gamma_0}$, in coordinates as in Section 3.2 (see the proof of Proposition 3.2). Suppose $\mu(\gamma) > 0$, so that the transition map $S^{in} \to S^{out}$ is defined.

Proposition A.1. *There exists $\tau_0 > 0$ so that for every $\tau > \tau_0$ and every $(\xi_{ss}, \xi_s, \xi_{uu}) \in E^{ss, s, uu}$ with $\|\xi\| \leqslant 1$, there is a unique orbit $(d/dt)\, y(t) = X_\gamma(y(t))$ with $y_{ss, s, c}(0) = (\xi_{ss}, \xi_s, -1)$ and $y_{uu}(\tau) = \xi_{uu}$.*

*Proof.* See [7], [8]. ∎

In the next two propositions, we treat properties of orbits $y(t)$ as in the above proposition. Such orbits start in $S^{in}$ but do not necessarily end up in $S^{out}$. However, the results below allow us to obtain sufficiently precise estimates for orbits that do end up in $S^{out}$, to prove Proposition 3.2. This requires some knowledge on the transition time from $S^{in}$ to $S^{out}$ (as function of $(\xi_{ss}, \xi_s, \xi_{uu})$ and of the parameter $\gamma$), which is treated in Proposition A.4.

The proposition below is similar to results in [9], [6], [24]. For completeness, and as a preparation for more precise asymptotics below, we give the main steps of the proof. With $\delta$ as in (3.22), (3.23), consider the rescaling $y_{ss, s, uu} \to y_{ss, s, uu}/\delta$. In the rescaled coordinates one has that $\|F^{ss}\|$, $\|F^s\|$ and $\|F^{uu}\|$ are bounded by $C\delta$, for some $C > 0$.

Proposition A.2. *For $\tau > \tau_0$ and $\xi_{ss, s, uu} \in E^{ss, s, uu}$ with $\|\xi_{ss, s, uu}\| \leqslant 1$, let $y(t, \tau, \xi; \gamma) = (y_{ss}, y_s, y_c, y_{uu})(t, \tau, \xi; \gamma)$ be the orbit of $X(\,\cdot\,; \gamma)$ with*

$$(y_{ss}, y_s, y_c)(0, \tau, \xi; \gamma) = (\xi_{ss}, \xi_s, -1),$$

$$y_{uu}(\tau, \tau, \xi; \gamma) = \xi_{uu}.$$

*Let $y_c^0(t, \tau; \gamma) = y_c(t, \tau, (0, 0, -1, 0); \gamma)$ and write*

$$z_c(t, \tau, \xi; \gamma) = y_c(t, \tau, \xi; \gamma) - y_c^0(t, \tau; \gamma).$$

*Let $\bar{v}^{uu}$ satisfy $0 < \bar{v}^{uu} < v^{uu}$, let $\bar{v}^{ss}$ satisfy $0 > \bar{v}^{ss} > \{2v^s, v^{ss}\}$ and let $\sigma^s = \max\{v^s, -\bar{v}^{uu}\}$. For $i \geqslant 0$, there are positive constants $C_i$ so that, for $0 \leqslant t \leqslant \tau$ and $\gamma$ near $\gamma_0$,*

$$\|D^k y_{ss}(t, \tau, \xi; \gamma)\| \leqslant C_k e^{\bar{v}^{ss} t},$$

$$\|D^k y_s(t, \tau, \xi; \gamma)\| \leqslant C_k e^{v^s t},$$

$$\|D^k z_c(t, \tau, \xi; \gamma)\| \leqslant C_k e^{\sigma^s t + \bar{v}^{uu}(t - \tau)},$$

$$\|D^k y_{uu}(t, \tau, \xi; \gamma)\| \leqslant C_k e^{\bar{v}^{uu}(t - \tau)},$$

*where $D^k$ stands for the $k^{\text{th}}$ order derivative in $(t, \xi, \gamma)$. Furthermore, for the derivatives with respect to $\tau$,*

$$\left\| D^k \frac{\partial}{\partial \tau} y_{ss}(t, \tau, \xi; \gamma) \right\| \leqslant C_k e^{\bar{\nu}^{ss} t + \bar{\nu}^{uu}(t-\tau)},$$

$$\left\| D^k \frac{\partial}{\partial \tau} y_s(t, \tau, \xi; \gamma) \right\| \leqslant C_k e^{\nu^s t + \bar{\nu}^{uu}(t-\tau)},$$

$$\left\| D^k \frac{\partial}{\partial \tau} z_c(t, \tau, \xi; \gamma) \right\| \leqslant C_k e^{\sigma^s t + \bar{\nu}^{uu}(t-\tau)},$$

$$\left\| D^k \frac{\partial}{\partial \tau} y_{uu}(t, \tau, \xi; \gamma) \right\| \leqslant C_k e^{\bar{\nu}^{uu}(t-\tau)},$$

*where $D^k$ stands for the $k^{\text{th}}$ order derivative in $(t, \tau, \xi, \gamma)$.*

*Proof.* We consider the estimates for $y_{ss,s}, z_c, y_{uu}$ first, then study asymptotics of their derivatives. To keep the notation simple, we write e.g. $y(t)$ for $y(t, \tau, \xi; \gamma)$.

By the variation of constants formula,

$$y_{ss}(t) = e^{B^{ss} t} \xi_{ss} + \int_0^t e^{B^{ss}(t-s)} F^{ss}(y(s))\, ds, \tag{A.57}$$

$$y_s(t) = e^{\int_0^t B^s(y_c(v))\, dv} \xi_s + \int_0^t e^{\int_s^t B^s(y_c(v))\, dv} F^s(y(s))\, ds, \tag{A.58}$$

$$z_c(t) = \int_0^t L(y_c^0(s), z_c(s))\, z_c(s) + F^c(y(s))\, ds, \tag{A.59}$$

$$y_{uu}(t) = e^{B^{uu}(\tau - t)} \xi_{uu} + \int_t^\tau e^{B^{uu}(t-s)} F^{uu}(y(s))\, ds, \tag{A.60}$$

where

$$L(y_c^0, z_c) = \int_0^1 \frac{\partial}{\partial y_c} U^c(y_c^0 + v z_c)\, dv.$$

Furthermore,

$$e^{\int_0^t B^s(y_c(v))\, dv} = e^{\nu^s t} e^{\int_0^t R^s(y_c(v))\, dv},$$

where

$$R^s(y_c) = \begin{pmatrix} 0 & -\omega^s(y_c) \\ \omega^s(y_c) & 0 \end{pmatrix}.$$

Note that $L = \mathcal{O}(\delta(|y_c^0| + |z_c|))$.

For $\alpha \leqslant 0$, $\beta \geqslant 0$ and a finite dimensional vector space $E$ with norm $\| \cdot \|$, let

$$\Sigma_{\alpha, \beta}([0, \tau], E) = \{ y \in C^0([0, \tau], E);\ \sup_{0 \leqslant t \leqslant \tau} \|y(t)\|\, e^{-\alpha t - \beta(t-\tau)} < \infty \}.$$

Equipped with the norm

$$\|y\|_{\alpha, \beta} = \sup_{0 \leqslant t \leqslant \tau} \|y(t)\|\, e^{-\alpha t - \beta(t-\tau)},$$

$\Sigma_{\alpha, \beta}([0, \tau], E)$ is a Banach space. Let

$$\begin{aligned} \mathbf{\Sigma} = {} & \Sigma_{\bar{\nu}^{ss}, 0}([0, \tau], E^{ss}) \times \Sigma_{\nu^s, 0}([0, \tau], E^s) \\ & \times \Sigma_{\sigma^s, \bar{\nu}^{uu}}([0, \tau], E^c) \times \Sigma_{0, \bar{\nu}^{uu}}([0, \tau], E^{uu}) \end{aligned}$$

and let $\mathbf{B}_R$ denote the ball of radius $R$ in $\mathbf{\Sigma}$.

Let $\mathfrak{Y} = (\mathfrak{Y}^{ss}, \mathfrak{Y}^s, \mathfrak{Y}^c, \mathfrak{Y}^{uu})$ be the map on $C^0([0, \tau], \mathbb{R}^n)$ that maps $(y_{ss}, y_s, z_c, y_{uu})$ to the right hand side of (A.57), (A.58), (A.59), (A.60). We claim that for $\|\xi_{ss, s, uu}\| \leqslant 1$, there exists $R > 0$ so that

- $\mathfrak{Y}$ maps $\mathbf{B}_R$ inside itself,
- $\mathfrak{Y}$ is a contraction on $\mathbf{B}_R$.

The fixed point of $\mathfrak{Y}$, providing the orbit $y$, therefore satisfies the estimates in the statement of the proposition.

We show that $\mathfrak{Y}$ maps $\mathbf{B}_R$ into itself. This follows from the following estimates. Below, $C$ denotes a constant which can change from one line to the next. With $y_s^2$ we mean quadratic terms in $y_s$. The estimates rely on (3.30), (3.31), (3.32), (3.33).

$$\begin{aligned} & \|e^{-\bar{\nu}^{ss} t} \mathfrak{Y}^{ss}(y_{ss, s}, z_c, y_{uu})(t)\| \\ & \quad = \left\| e^{(B^{ss} - \bar{\nu}^{ss} I)\, t} \xi_{ss} + \int_0^t e^{(B^{ss} - \bar{\nu}^{ss} I)(t-s)} e^{-\bar{\nu}^{ss} s} F_{ss}(y(s); \gamma)\, ds \right\| \\ & \quad \leqslant C \|\xi_{ss}\| + C \sup \|F_{ss}\| \, (\|y_{ss}\|_{\bar{\nu}^{ss}, 0} + \|y_s^2\|_{\bar{\nu}^{ss}, 0}) \\ & \quad \leqslant C \|\xi_{ss}\| + C\delta(\|y_{ss}\|_{\bar{\nu}^{ss}, 0} + \|y_s^2\|_{\bar{\nu}^{ss}, 0}), \end{aligned}$$

$$\begin{aligned}
&\|e^{-\nu^s t}\mathfrak{Y}^s(y_{ss,s}, z_c, y_{uu})(t)\| \\
&\quad = \left\| e^{\int_0^t R^s(y_c(v))\,dv}\xi_s + \int_0^t e^{-\nu^s s} e^{\int_s^t R^s(y_c(v))\,dv} F^s(y(s))\,ds \right\| \\
&\quad \leqslant \|\xi_s\| + \int_0^t \|e^{-\nu^s s}F^s(y(s))\|\,ds \\
&\quad \leqslant \|\xi_s\| + C\delta\,\|y_{ss,s}\|_{\nu^s,0}, \\
&\|e^{-\sigma^s t + \bar{\nu}^{uu}(t-\tau)}\mathfrak{Y}^c(y_{ss,s}, z_c, y_{uu})(t)\| \\
&\quad = \left\| e^{-\sigma^s t + \bar{\nu}^{uu}(t-\tau)} \int_0^t L(y_c^0(s), z_c(s))\,z_c(s) + F^c(y(s))\,ds \right\| \\
&\quad \leqslant \sup |L|\,\|z_c\|_{\sigma^s, \bar{\nu}^{uu}} + \sup \|F^c\|\ \|y_{ss,s}\|_{\sigma^s,0}\,\|y_{uu}\|_{0,\bar{\nu}^{uu}} \\
&\quad \leqslant \delta\,\|z_c\|_{\sigma^s,\bar{\nu}^{uu}} + \delta\,\|y_{ss,s}\|_{\sigma^s,0}\,\|y_{uu}\|_{0,\bar{\nu}^{uu}}, \\
&\|e^{\bar{\nu}^{uu}(t-\tau)}\mathfrak{Y}^{uu}(y_{ss,s}, z_c, y_{uu})(t)\| \\
&\quad \leqslant C\,\|\xi_{uu}\| + C\delta\,\|y_{uu}\|_{0,\bar{\nu}^{uu}}.
\end{aligned}$$

We conclude that, for $\delta$ small enough, there exists $R > 0$ so that $\mathfrak{Y}$ maps $B_R$ inside itself.

The proof that $\mathfrak{Y}$ is a contraction on $B_R$, proceeds similarly. Let $y^1$ and $y^2$ be two orbits of $X$. Then e.g.

$$\begin{aligned}
&\|e^{-\bar{\nu}^{ss}t}(\mathfrak{Y}^{ss}(y^1_{ss,s}, z^1_c, y^1_{uu})(t)) - \mathfrak{Y}^{ss}(y^2_{ss,s}, z^2_c, y^2_{uu})(t))\| \\
&\quad \leqslant C\,\sup\|F_{ss}(y)\|(\|y^1_{ss} - y^2_{ss}\|_{\bar{\nu}^{ss},0} + \|y^1_s - y^2_s\|_{\nu^s,0}) \\
&\quad \leqslant C\delta(\|y^1_{ss} - y^2_{ss}\|_{\bar{\nu}^{ss},0} + \|y^1_s - y^2_s\|_{\nu^s,0}).
\end{aligned}$$

With similar estimates for $\mathfrak{Y}^s$, $\mathfrak{Y}^c$, $\mathfrak{Y}^{uu}$, this implies that $\mathfrak{Y}$ is a contraction on $B_R$.

One treats (higher order) derivatives by differentiating (A.57), (A.58), (A.59), (A.60) and using the obtained identities to define a map on an appropriate weighted Banach space. Performing estimates as above one shows that this map is a contraction on some ball in the weighted Banach space. For details we refer to [7]. ▮

To derive appropriate bifurcation equations, we need more precise information on the asymptotics of $y$. This is the contents of the following proposition, where precise asymptotics for $y_s$ is obtained. This proposition is the analogue of asymptotic expansions as derived in [7] for Shil'nikov variables near hyperbolic singularities. The proof is an adaptation of the computations in [7], [8], [9].

PROPOSITION A.3. *For $\tau > \tau_0$ and $\xi_{ss,\,s,\,uu} \in E^{ss,\,s,\,uu}$ with $\|\xi_{ss,\,s,\,uu}\| \leqslant 1$, let $y(t, \tau, \xi; \gamma) = (y_{ss}, y_s, y_c, y_{uu})(t, \tau, \xi; \gamma)$ be the orbit of $X(\,\cdot\,; \gamma)$ with*

$$(y_{ss}, y_s, y_c)(0, \tau, \xi; \gamma) = (\xi_{ss}, \xi_s, -1),$$

$$y_{uu}(\tau, \tau, \xi; \gamma) = \xi_{uu}.$$

*For $\bar{v}^{ss} < \hat{v}^{ss} < v^s$, let*

$$z_s(u, \tau, \xi; \gamma) = e^{-\int_u^\tau B^s(y_c(v))\,dv} y_s(\tau - u, \tau, \xi; \gamma),$$

*Then $z_s^\infty(u, \xi; \gamma) = \lim_{\tau \to \infty} z_s(u, \tau, \xi; \gamma)$ exists and there are $\sigma > 0$, $C_k > 0$ so that*

$$\|D^k(z_s(u, \tau, \xi; \gamma) - z_s^\infty(u, \xi; \gamma))\| \leqslant C_k e^{\sigma(u - \tau)},$$

*where $D^k$ stands for the $k^{\text{th}}$ order derivative in $(u, \tau, \xi, \gamma)$.*

*Proof.* We will first show that

$$\left\| \frac{\partial}{\partial \tau} z_s(u, \tau, \xi; \gamma) \right\| \leqslant C e^{\sigma(u - \tau)}, \tag{A.61}$$

for some $C, \sigma > 0$. From this it follows that $z_s^\infty(u, \xi; \gamma) = \lim_{\tau \to \infty} z_s(u, \tau, \xi; \gamma)$ exists and

$$\|z_s(u, \tau, \xi; \gamma) - z_s^\infty(u, \xi; \gamma)\| \leqslant C e^{-\sigma(\tau - u)}.$$

The strategy for showing (A.61) is the same as in the proof of Proposition A.2; we will construct an appropriate contraction on a weighted Banach space. As in the proof of Proposition A.2, we simplify the notation and write e.g. $y(t)$ for $y(t, \tau, \xi; \gamma)$.

We have

$$z_s(u) = \xi_s + \int_0^{\tau - u} e^{-v^s s} e^{-\int_0^s R^s(y_c(v))\,dv} F^s(y(s))\,ds. \tag{A.62}$$

Write

$$F^s(y) = y_{ss,\,s} y_{uu} h^{uu}(y) + y_{ss,\,s} y_{ss,\,s,\,c} h^{ss,\,s,\,c}(y).$$

By splitting the integral in (A.62) according to the terms occurring in the sums and changing the integration parameter to $\alpha = \tau - s$ in the integrals containing $h^{uu}$, we get

$$z_s(u) = \xi_s + \int_0^{\tau - u} e^{-\nu^s s} e^{-\int_0^s R^s(y_c(v))\, dv} y_{ss,\,s}(s)\, y_{ss,\,s}(s)\, h^{ss,\,s,\,c}(y(s))\, ds$$

$$+ \int_u^{\tau} e^{-\nu^s(\tau - \alpha)} e^{-\int_0^{\tau - \alpha} R^s(y_c(v))\, dv} \bar{y}_{ss,\,s}(\alpha)\, \bar{y}_{uu}(\alpha)\, h^{uu}(\bar{y}(\alpha))\, d\alpha,$$

where $\bar{y}(\alpha) = y(\tau - \alpha)$. Note that $\bar{y}$ is an orbit of $-X$ with $\bar{y}_{ss}(\tau) = \xi_{ss}$, $\bar{y}_s(\tau) = \xi_s$ and $\bar{y}_{uu}(0) = \xi_{uu}$. An appropriately altered version of Proposition A.2 can therefore be applied to $\bar{y}$. Indeed, the estimates in Proposition A.2 also hold if one replaces the condition $y_c(0) = -1$ by $y_c(\tau) = 1$. In particular, for $\bar{y}$ one obtains estimates of the following form, for some $C > 0$:

$$\|\bar{y}_{ss}(\alpha)\| \leqslant Ce^{-\bar{\nu}^{ss}(\alpha - \tau)},$$

$$\|\bar{y}_s(\alpha)\| \leqslant Ce^{-\nu^s(\alpha - \tau)},$$

$$\|\bar{z}_c(\alpha)\| \leqslant Ce^{-\sigma^s(\alpha - \tau) - \bar{\nu}^{uu}\alpha},$$

$$\|\bar{y}_{uu}(\alpha)\| \leqslant Ce^{-\bar{\nu}^{uu}\alpha}$$

and for derivatives with respect to $\tau$,

$$\left\|\frac{\partial}{\partial\tau}\, \bar{y}_{ss}(\alpha)\right\| \leqslant Ce^{-\bar{\nu}^{ss}(\alpha - \tau)},$$

$$\left\|\frac{\partial}{\partial\tau}\, \bar{y}_s(\alpha)\right\| \leqslant Ce^{-\nu^s(\alpha - \tau)},$$

$$\left\|\frac{\partial}{\partial\tau}\, \bar{z}_c(\alpha)\right\| \leqslant Ce^{-\sigma^s(\alpha - \tau) - \bar{\nu}^{uu}\alpha},$$

$$\left\|\frac{\partial}{\partial\tau}\, \bar{y}_{uu}(\alpha)\right\| \leqslant Ce^{-\nu^s(\alpha - \tau) - \bar{\nu}^{uu}\alpha}.$$

Compute

$$\frac{\partial}{\partial\tau}\, z_s(u, \tau) = T_1 + T_2 + I_1 + I_2 + I_3 + I_4, \tag{A.63}$$

where

$$T_1 = e^{-\nu^s(\tau - u)} e^{-\int_0^{\tau - u} R^s(y_c(v))\, dv} y_{ss,\,s}(\tau - u)\, y_{ss,\,s}(\tau - u)\, h^{ss,\,s}(y(\tau - u)),$$

$$T_2 = \bar{y}_{ss,\,s}(\tau)\, \bar{y}_{uu}(\tau)\, h^{uu}(\bar{y}(\tau)),$$

and

$$I_1 = \int_0^{\tau - u} e^{-\nu^s s} \frac{\partial}{\partial \tau} [e^{-\int_0^s R^s(y_c(v))\,dv} y_{ss,\,s}(s)\, y_{ss,\,s}(s)\, h^{ss,\,s}(y(s))]\, ds,$$

$$I_2 = \int_u^{\tau} \frac{\partial}{\partial \tau} z_s(\alpha)\, \bar{y}_{uu}(\alpha)\, h_s^{uu}(\bar{y}(\alpha))\, d\alpha,$$

$$I_3 = \int_u^{\tau} e^{-\nu^s(\tau - \alpha)} e^{-\int_0^{\tau-\alpha} R^s(y_c(v))\,dv} \bar{y}_s(\alpha) \frac{\partial}{\partial \tau} [\bar{y}_{uu}(\alpha)\, h_s^{uu}(\bar{y}(\alpha))]\, d\alpha,$$

$$I_4 = \int_u^{\tau} \frac{\partial}{\partial \tau} e^{-\nu^s(\tau - \alpha)} e^{-\int_0^{\tau-\alpha} R^s(y_c(v))\,dv} \bar{y}_{ss}(\alpha)\, \bar{y}_{uu}(\alpha)\, h_{ss}^{uu}(\bar{y}(\alpha))\, d\alpha.$$

In $I_2$, $I_3$ and $I_4$ we have written

$$y_{ss,\,s}\, y_{uu} h^{uu}(y) = y_{ss}\, y_{uu} h_{ss}^{uu}(y) + y_s\, y_{uu} h_s^{uu}(y).$$

Let $\mathfrak{Z}$ be the map on $C^0([0, \tau], E^s)$ that maps $z_s$ to the right hand side of (A.63). Let $B_R([0, \tau], E^s)$ be the ball of radius $R$ in $\Sigma_{\sigma,0}([0, \tau], E^s)$. We claim that, for some $\sigma > 0$ and $R > 0$,

- $\mathfrak{Z}$ maps $B_R([0, \tau], E^s)$ into itself,
- $\mathfrak{Z}$ is a contraction on $B_R([0, \tau], E^s)$.

The estimates are very similar to the ones in the proof of Proposition A.2; the particular way of composing $\mathfrak{Z}$ in the above sum of terms enables direct estimates using Proposition A.2. In the estimates one uses $(\partial/\partial\tau)\, y_c(t) = (\partial/\partial\tau)\, z_c(t)$, which follows from $(\partial/\partial\tau)\, y_c^0(t) = 0$. We will show that $\mathfrak{Z}$ maps $B_R([0, \tau], E^s)$ into itself, leaving the similar estimates showing that $\mathfrak{Z}$ is actually a contraction to the reader. Compare also the proof of Proposition 6.2. In the following, $C$ denotes a constant, that may change from line to line, but is uniformly bounded. The claim that $\mathfrak{Z}$ maps $B_R([0, \tau], E^s)$ into itself follow from the next list of estimates. In its derivation we use Proposition 6.2.

$$\begin{aligned}
|T_1| &\leqslant e^{-\nu^s(\tau - u)} \|y_{ss,\,s}(\tau - u)\|\ \|y_{ss,\,s}(\tau - u)\|\ \|h^{ss,\,s}(y(\tau - u))\| \\
&\leqslant C\delta e^{-\nu^s(\tau - u)} e^{2\nu^s(\tau - u)} \\
&\leqslant C\delta e^{\nu^s(\tau - u)},
\end{aligned}$$

$$\begin{aligned}
|T_2| &\leqslant \|\bar{y}_{ss,\,s}(\tau)\|\ \|\bar{y}_{uu}(\tau)\|\ \|h^{uu}(\bar{y}(\tau))\| \\
&\leqslant C\delta e^{-\nu^{uu}\tau} \\
&\leqslant C\delta e^{-\nu^{uu}(\tau - u)}.
\end{aligned}$$

Similarly one bounds the integrals $I_1, ..., I_4$.

$$|I_1| \leqslant \int_0^{\tau-u} Ce^{-\nu^s s} \left| \frac{\partial}{\partial \tau} \left[ e^{-\int_0^s R^s(y_c(v))\, dv} y_{ss,s}(s)\, y_{ss,s}(s)\, h^{ss,s}(y(s)) \right] \right| ds,$$

$$\leqslant \int_0^{\tau-u} C\delta e^{-\nu^s s} (e^{\nu^s s} e^{\nu^s s + \bar{\nu}^{uu}(s-\tau)} + e^{\nu^s s} e^{\sigma^s s - \bar{\nu}^{uu}(s-\tau)})\, ds$$

$$\leqslant C\delta (e^{\nu^s(\tau-u)} + e^{-\bar{\nu}^{uu}\tau} + e^{\sigma^s(\tau-u)})$$

$$\leqslant C\delta (e^{\nu^s(\tau-u)} + e^{-\bar{\nu}^{uu}(\tau-u)} + e^{\sigma^s(\tau-u)}),$$

$$|I_2| \leqslant \int_u^{\tau} \left| \frac{\partial}{\partial \tau} z_s(\alpha) \right| \|\bar{y}_{uu}(\alpha)\|\ \|h_s^{uu}(\bar{y}(\alpha))\| d\alpha$$

$$\leqslant \int_u^{\tau} C\delta \left| \frac{\partial}{\partial \tau} z_s(\alpha) \right| e^{-\bar{\nu}^{uu}\alpha}\, d\alpha$$

$$\leqslant C\delta \left( e^{-\bar{\nu}^{uu}\tau} + \left| \frac{\partial}{\partial \tau} z_s(u) \right| \right)$$

$$\leqslant C\delta \left( e^{-\bar{\nu}^{uu}(\tau-u)} + \left| \frac{\partial}{\partial \tau} z_s(u) \right| \right),$$

$$|I_3| \leqslant \int_u^{\tau} e^{-\nu^s(\tau-\alpha)} \|\bar{y}_s(\alpha)\| \left| \frac{\partial}{\partial \tau} [\bar{y}_{uu}(\alpha)\, h_s^{uu}(\bar{y}(\alpha))] \right| d\alpha$$

$$\leqslant \int_u^{\tau} C\delta e^{-\nu^s(\tau-\alpha)} e^{-\nu^s(\alpha-\tau)} (e^{-\nu^s(\alpha-\tau) - \bar{\nu}^{uu}\alpha} + e^{-\bar{\nu}^{uu}\alpha} e^{-\sigma^s(\alpha-\tau) - \bar{\nu}^{uu}\alpha})\, d\alpha$$

$$\leqslant C\delta (e^{-\bar{\nu}^{uu}\tau} + e^{-\nu^s(u-\tau)})$$

$$\leqslant C\delta (e^{-\bar{\nu}^{uu}(\tau-u)} + e^{\nu^s(\tau-u)}),$$

$$|I_4| \leqslant \int_u^{\tau} \left| \frac{\partial}{\partial \tau} e^{-\nu^s(\tau-\alpha)} \right| \|\bar{y}_{ss}(\alpha)\|\ \|\bar{y}_{uu}(\alpha)\|\ \|h_{ss}^{uu}(\bar{y}(\alpha))\|\, d\alpha$$

$$\leqslant \int_u^{\tau} C\delta e^{-\nu^s(\tau-\alpha)} e^{-\bar{\nu}^{ss}(\alpha-\tau)} e^{-\bar{\nu}^{uu}\alpha}\, d\alpha$$

$$\leqslant C\delta (e^{-\bar{\nu}^{uu}\tau} + e^{-\nu^s(\tau-u)} e^{-\bar{\nu}^{ss}(u-\tau)})$$

$$\leqslant C\delta (e^{-\bar{\nu}^{uu}(\tau-u)} + e^{(\bar{\nu}^{ss} - \nu^s)(\tau-u)}).$$

The uniqueness of the fixed point of $\mathfrak{Z}$ shows that the resulting function $z_s$ is indeed given by the identity in the statement of the proposition. Finally, for derivatives one uses the same approach as in the proof of Proposition A.2. ∎

The above results treat orbits $y(t)$ with the boundary conditions $y_{ss,s,c}(0) = (\xi_{ss}, \xi_s, -1)$ and $y_{uu}(\tau) = \xi_{uu}$ for $(\xi_{ss}, \xi_s, \xi_{uu}) \in E^{ss,s,uu}$ with

$\|(\xi_{ss}, \xi_s, \xi_{uu})\| \leqslant 1$ and $\tau > 0$ large enough. We need however statements on orbits $y(t)$ that go from $S^{in}$ to $S^{out}$. This defines $\tau$ as function of $(\xi_{ss}, \xi_s, \xi_{uu})$ and $\gamma$.

PROPOSITION A.4. *For $\xi_{ss,\,s,\,uu} \in E^{ss,\,s,\,uu}$ with $\|\xi_{ss,\,s,\,uu}\| \leqslant 1$, let $\tau = \hat{\tau}(\xi_{ss,\,s,\,uu}; \gamma)$ be such that $y(t, \tau, \xi; \gamma) = (y_{ss}, y_s, y_c, y_{uu})(t, \tau, \xi; \gamma)$ is the orbit of $X(\,\cdot\,; \gamma)$ with*

$$(y_{ss}, y_s, y_c)(0, \tau, \xi; \gamma) = (\xi_{ss}, \xi_s, -1),$$

$$(y_c, y_{uu})(\tau, \tau, \xi; \gamma) = (1, \xi_{uu}).$$

*Then $\hat{\tau}$ is a smooth function of $\xi_{ss,\,s,\,uu}$ and $\gamma$, if $\mu(\gamma) > 0$. For some $C^k > 0$, it satisfies*

$$\left\| \frac{\partial^k}{\partial \gamma^k} \hat{\tau}(0; \gamma) \right\| \leqslant C_k\, \mu(\gamma)^{(1/2) - k}.$$

*Furthermore, for some $C_k > 0$, $\sigma < 0$,*

$$\|D^k(\hat{\tau}(\xi_{ss,\,s,\,uu}; \gamma) - \hat{\tau}(0; \gamma))\| \leqslant C_k e^{\sigma \hat{\tau}(0;\, \gamma)}, \tag{6.34}$$

*where $D^k$ stands for the $k^{\text{th}}$ order derivative in $(\xi, \gamma)$. In the above, $C_0 = \mathcal{O}(\delta)$.*

*Remark.* Let $\rho = e^{-\hat{\tau}(0;\, \gamma)}$. It follows from the estimates on $\hat{\tau}(0; \gamma)$, that $\rho$ and its derivatives are flat functions, as $\mu(\gamma) \to 0$.

*Proof.* From Proposition A.2 it is clear that $\hat{\tau}$ is a smooth function of $\xi$ and $\gamma$. Note that

$$\hat{\tau}(0; \gamma) = \int_{-1}^{1} \frac{1}{U^c(w; \gamma)}\, dw,$$

where $U^c$ is as in (3.35). For the estimates on $\hat{\tau}(0; \gamma)$, see [12]. The remaining estimates on $\hat{\tau}(\xi_{ss,\,s,\,uu}; \gamma) - \hat{\tau}(0; \gamma)$ follow from these estimates and Proposition A.2. ∎

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